

Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024

Prof. Dr. Bernd Ammann, Julian Seipel

Please hand in the exercises until **Tuesday, April 23, 12:00 in the letterbox no. 16.**



Exercise Sheet no. 1

1. Exercise (4 points).

Assume that $(M, g), (N, h)$ are surfaces with Riemannian metrics with negative Gauß curvature. Is it true that the product manifold $(M \times N, g + h)$ has everywhere negative sectional curvature?

2. Exercise (4 points).

Consider the following subsets of $\mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2}$:

$$\begin{aligned} \mathrm{SO}(n) &= \{A \in \mathbb{R}^{n \times n} \mid A^T A = \mathbf{1}, \det_{\mathbb{R}}(A) = 1\} \\ \mathrm{GL}(m, \mathbb{C}) &= \{A \in \mathbb{R}^{2m \times 2m} \mid AJ = JA, \det_{\mathbb{R}}(A) \neq 0\}, \quad \text{if } m = n/2 \in \mathbb{N} \\ \mathrm{U}(m) &= \{A \in \mathrm{GL}(2m, \mathbb{C}) \mid A^* A = \mathbf{1}\} \\ \mathrm{SU}(m) &= \{A \in \mathrm{GL}(2m, \mathbb{C}) \mid A^* A = \mathbf{1}, \det_{\mathbb{C}}(A) = 1\} \\ \mathrm{Aff}(n) &= \left\{ \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(k+1, \mathbb{R}) \mid A \in \mathrm{GL}(k, \mathbb{R}), b \in \mathbb{R}^k \right\}, \quad k = n-1 \end{aligned}$$

where we used in for $m = n/2 \in \mathbb{N}$ the definition $J := \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \in \mathbb{R}^{2m \times 2m}$.

- For a matrix $A \in \mathbb{R}^{2m \times 2m}$ with $AJ = JA$ give a reasonable definition for the “complex determinant” $\det_{\mathbb{C}} A$, and show that $\det_{\mathbb{C}} A \neq 0$ if and only if $\det_{\mathbb{R}} A \neq 0$. (Bonus exercise: derive a formula for $\det_{\mathbb{R}} A$ in terms of $\det_{\mathbb{C}} A$.)
- Show that these are Lie groups with the manifold structure induced from $\mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2}$.
- Determine their Lie algebras (i.e. as linear subspaces of $\mathbb{R}^{n \times n}$). What is the Lie bracket?
- Construct a Lie algebra isomorphism (i.e. an isomorphism of vector spaces preserving the Lie bracket) between the Lie algebra $\mathfrak{so}(3)$ and the Lie algebra (\mathbb{R}^3, \times) , where \times denotes the cross product.

In the following we define the adjoint map of a Lie group G with Lie algebra \mathfrak{g} as

$$\mathrm{Ad}_{\mathfrak{g}} := d_{\mathbf{1}} C_g : \mathfrak{g} \rightarrow \mathfrak{g}$$

3. Exercise (4 points).

Let G be a Lie group and \mathfrak{g} its Lie algebra and let $g \in G$. We denote with ℓ_g (respectively r_g) the left multiplication by g (respectively right multiplication by g) of the Lie group. A Riemannian metric $\langle \cdot, \cdot \rangle$ on G is called *left-invariant* (resp. *right-invariant*) if ℓ_g (resp. r_g) is an isometry of $\langle \cdot, \cdot \rangle$ for all $g \in G$. A metric is called *bi-invariant metric* if it is right- and left-invariant. Show:

- a) A scalar product on the Lie algebra \mathfrak{g} can be extended uniquely to a left-invariant metric on G . The same holds if we replace “left-invariant” by “right-invariant”.
- b) A left-invariant metric $\langle \cdot, \cdot \rangle$ is bi-invariant iff its restriction $\langle \cdot, \cdot \rangle_{\mathbb{1}} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ is Ad-invariant metric, i.e. if it is invariant under pullback by Ad_g for all $g \in G$.
- c) The bilinear map $(A, B) \mapsto -\text{tr}_{\mathbb{R}} AB$ defines an Ad-invariant metric on $\mathfrak{so}(n)$, $\mathfrak{u}(m)$, and $\mathfrak{su}(m)$, which are the Lie algebras of the corresponding groups in Exercise 2.
- d) If the Lie group is compact, then there exists a bi-invariant metric on this Lie group.
Hint: Take a left-invariant metric $\langle \cdot, \cdot \rangle$ and show that

$$\langle X, Y \rangle'_h := \int_G \langle d\ell_g X, d\ell_g Y \rangle_{gh} \text{dvol}(g)$$

for $X, Y \in T_{\mathbb{1}}G$, $h \in G$ and dvol a right-invariant volume form. This gives a bi-invariant metric on G .

4. Exercise (4 points).

Let $(V, [\cdot, \cdot]_V)$ be a Lie algebra over a field \mathbb{K} . A linear subspace $W \subset V$ is called an *ideal* if for all $X \in W, Y \in V$ we have $[X, Y] \in W$. Show:

- a) The quotient vector space V/W carries a unique Lie bracket $[\cdot, \cdot]_{V/W}$ such that the quotient map $\pi: V \rightarrow V/W$ is a Lie algebra homomorphism.
- b) The kernel of a Lie algebra homomorphism is an ideal. Moreover every ideal of a Lie algebra is the kernel of a Lie algebra homomorphism.
- c) Let $\mathbb{K} = \mathbb{R}$ and G, H Lie groups with $H \triangleleft G$ be a normal subgroup and submanifold. The Lie algebra of H is an ideal of the Lie algebra of G .

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Please hand in the exercises until **Tuesday, April 30, 12:00 in the letterbox no. 16.**



Exercise Sheet no. 2

1. Exercise (4 points).

Let G be a Lie group and \mathfrak{g} its Lie algebra. We recall:

$$c: G \times G \rightarrow G, (g, h) \mapsto ghg^{-1}, \quad (\text{conjugation})$$

$$\text{Ad}: G \rightarrow \text{Aut}(\mathfrak{g}), g \mapsto d_1 c(g, \cdot), \quad (\text{adjoint representation})$$

$$\text{ad} = d_1 \text{Ad}: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}), \quad (\text{adjoint map}).$$

Determine all these maps explicitly in the case of a matrix group $G \subset \mathbb{R}^{n \times n}$. In particular show $\text{ad}_X(Y) = [X, Y]$.

2. Exercise (4 points).

Let G and H be two Lie groups and $\varphi: G \rightarrow H$ is a Lie group homomorphism. Show:

- The differential $d_1 \varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ is surjective if and only if φ is a submersion.
- The differential $d_1 \varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ is bijective if and only if φ is a local diffeomorphism.
- Assume that H is connected and the differential $d_1 \varphi: \mathfrak{g} \rightarrow \mathfrak{h}$ is surjective. Show that φ is surjective. *Hint: Show that the image of the map φ is closed and open.*
- If $\varphi: G \rightarrow H$ is a local diffeomorphism, then it is a covering map.

3. Exercise (4 points).

Show that there does not exist a matrix $A \in \mathfrak{gl}(2, \mathbb{R})$, such that $\exp(A) = \begin{pmatrix} -\pi & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$ holds. Deduce a contradiction by considering the following cases:

- A is diagonalizable.
- A is triangularizable, but not diagonalizable.
- A has no real eigenvalues.

Moreover show that the following maps are well-defined:

$$\exp: \mathfrak{u}(n) \rightarrow \text{U}(n), \quad (1)$$

$$\exp: \mathfrak{so}(n) \rightarrow \text{SO}(n), \quad (2)$$

$$\exp: \text{Der}(\mathfrak{g}) \rightarrow \text{Aut}(\mathfrak{g}),$$

where $\text{Der}(\mathfrak{g})$ denotes the derivation of a Lie algebra \mathfrak{g} . Show that the maps (1) and (2) are surjective.

4. Exercise (4 points).

Let G be a Lie group and \mathfrak{g} its Lie algebra.

- a) Let $\langle \cdot, \cdot \rangle$ be a bi-invariant metric on G . Show that

$$\langle \mathbf{ad}_\xi \eta, \zeta \rangle + \langle \eta, \mathbf{ad}_\xi \zeta \rangle = 0$$

holds for all $\xi, \eta, \zeta \in \mathfrak{g}$.

- b) Show that the Levi-Civita connection of a bi-invariant metric on G is given by

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

for all left-invariant vector fields X, Y on G . *Hint: Use the Koszul formula. You may use that $\mathbf{ad}_\xi \eta = [\xi, \eta]$ for all $\xi, \eta \in \mathfrak{g}$.*

- c) Show that the geodesics of a bi-invariant metric on G passing through $\mathbf{1} \in G$ are exactly the 1-parameter subgroups of G .
- d) Conclude from the previous statements that the exponential map for a connected, compact Lie group is surjective.

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Please hand in the exercises until **Tuesday, May 7, 12:00 in the letterbox no. 16.**



Exercise Sheet no. 3

1. Exercise (4 points).

Let $\mathrm{SL}(2, \mathbb{R})$ be the Lie group of matrices of determinant 1 and its Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. Moreover let $\exp: \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathrm{SL}(2, \mathbb{R})$ be the exponential map and let $k(x) = \frac{1}{2} \mathrm{tr}(x^2)$ for an element $x \in \mathfrak{sl}(2, \mathbb{R})$. As proven in the lecture, the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is given by matrices of vanishing trace. Show:

a) The exponential is given by

$$\exp(x) = C(k(x)) \mathbb{1}_2 + S(k(x))x$$

for all $x \in \mathfrak{sl}(2, \mathbb{R})$. The functions $C, S: \mathbb{R} \rightarrow \mathbb{R}$ are given by

$$C(t) = \begin{cases} \cosh(\sqrt{t}) & \text{for } t \geq 0 \\ \cos(\sqrt{-t}) & \text{for } t < 0 \end{cases} \quad S(t) = \begin{cases} \frac{\sinh(\sqrt{t})}{\sqrt{t}} & \text{for } t > 0 \\ 1 & \text{for } t = 0 \\ \frac{\sin(\sqrt{-t})}{\sqrt{-t}} & \text{for } t < 0 \end{cases}$$

b) A 1-parameter subgroup $t \mapsto \exp(tx)$ of $\mathrm{SL}(2, \mathbb{R})$ has a compact image if and only if $k(x) < 0$ holds.

2. Exercise (4 points).

Let G be a Lie group which acts on a manifold M . Show that the following statements are equivalent:

- i) G acts *proper* on M , i.e. the shear map $\mathrm{shear}: G \times M \rightarrow M \times M, (g, x) \mapsto (gx, x)$ satisfies that preimages of compact subsets are compact again.
- ii) For all compact subsets $K \subset M$ the set $G_K := \{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is compact.
- iii) Let $(p_i)_{i \in \mathbb{N}}$ be a sequence in M and $(g_i)_{i \in \mathbb{N}}$ a sequence in G such that the sequences $(p_i)_{i \in \mathbb{N}}$ and $(g_i \cdot p_i)_{i \in \mathbb{N}}$ converge. Then we find a convergent subsequence of $(g_i)_{i \in \mathbb{N}}$.

3. Exercise (4 points).

Let G be a discrete group which acts on a manifold M . We call the group action *properly discontinuous* if for all points p and q in M there exist neighbourhoods U_p and U_q such that

$$\forall g \in G: (p \neq g \cdot q \Rightarrow U_p \cap (g \cdot U_q) = \emptyset)$$

holds.

a) Assume moreover that the group action is free. Show that the group action is proper if and only if the action is properly discontinuous.

b) Consider the group action

$$\begin{aligned}\mathbb{Z} \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ (k, x) &\mapsto 2^k x.\end{aligned}$$

Is this action properly discontinuous on $M_1 = \mathbb{R}^n$ or $M_2 = \mathbb{R}^n \setminus \{0\}$ or $M_3 = (0, \infty) \times (0, \infty) \times \mathbb{R}^{n-2}$?

c) We write $p_i: M_i \rightarrow M_i/\mathbb{Z}$ for the canonical maps. We equip each M_i/\mathbb{Z} with the quotient topology (i.e. $U \subset M_i/\mathbb{Z}$ is open in M_i/\mathbb{Z} iff $p_i^{-1}(U)$ is open in M_i). Are the quotients M_i/\mathbb{Z} Hausdorff or compact for $i = 1, 2, 3$?

4. Exercise (4 points).

We define the *3-dimensional Heisenberg group* H_3 as

$$H_3 := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

This is a submanifold and subgroup of $GL(3, \mathbb{R})$, thus a Lie group.

a) Show that the Lie algebra \mathfrak{h}_3 , the *3-dimensional Heisenberg Lie algebra*, is given by matrices as follows:

$$\mathfrak{h}_3 := \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$$

b) Calculate $\exp: \mathfrak{h}_3 \rightarrow H_3$, and show that it is a diffeomorphism.

c) Show that $\log(\exp(A)\exp(B)) = A + B + \frac{1}{2}[A, B]$ for all $A, B \in \mathfrak{h}_3$, where \log is the inverse function of \exp .

d) Show that \mathfrak{h}_3 is 2-step nilpotent, i.e. $[A, [B, C]] = 0$ for $A, B, C \in \mathfrak{h}_3$.

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This sheet will have to be solved **independently** in the 20th calendar week.



Exercise Sheet no. 4

1. Exercise (0 points).

Let G be a connected topological group.

- a) Let $\gamma_1, \gamma_2: [0, 1] \rightarrow G$ be two paths in G with $\gamma_1(0) = \gamma_2(0) = \gamma_1(1) = \gamma_2(1) = \mathbf{1}$. Show the two concatenations

$$\gamma_1 * \gamma_2 := \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2 \\ \gamma_2(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

and $\gamma_2 * \gamma_1$ defined analogously are homotopic with fixed endpoints. *Hint: To construct a homotopy, try modifications of the map $(t, s) \mapsto \gamma_1(t) \cdot \gamma_2(s+t)$ where we define $\gamma_1(t) = \gamma_2(t) = \mathbf{1}$ for $t \notin [0, 1]$.*

- b) Deduce that the fundamental group $\pi_1(G)$ is abelian.
c) Let Γ be a discrete normal subgroup of the Lie group H . Then Γ is abelian.

Hint: you may prove c) directly, and you may use c) to get b). Or you may apply b) to $G := \Gamma \backslash H$ and argue that covering space theory then provides a surjective homomorphism $\pi_1(G) \rightarrow \Gamma$ and then get c).

2. Exercise (0 points).

We consider the two Lie groups

$$G = \left\{ g(t, a, b) := \begin{pmatrix} \cos(t) & -\sin(t) & a \\ \sin(t) & \cos(t) & b \\ 0 & 0 & 1 \end{pmatrix} \mid t, a, b \in \mathbb{R} \right\}$$
$$\tilde{G} = \left\{ \tilde{g}(t, a, b) := \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) & a \\ 0 & 0 & \sin(t) & \cos(t) & b \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mid t, a, b \in \mathbb{R} \right\}$$

Show:

- a) The projection $\pi: \tilde{G} \rightarrow G$ $\tilde{g}(t, a, b) \rightarrow g(t, a, b)$ is the universal covering of G .
b) The exponential map for G is surjective.
c) The exponential map for \tilde{G} is not surjective.

3. Exercise (0 points).

Let G be topological group and denote by l_g (respectively r_g) the left (right) multiplication by a group element g on the group.

- a) Show that an automorphism $\varphi: G \rightarrow G$ of the topological group, which commutes with all right multiplication, is given by a left multiplication.
- b) Assume that G is a compact Lie group and $H \subset G$ is a subgroup and a submanifold. If the subgroup H is isomorphic to G as an abstract group, then $H = G$.

4. Exercise (0 points).

Let A be a connected abelian Lie group. Show that:

- a) The exponential map \exp_A is surjective.
- b) The kernel $\Gamma_A := \ker(\exp_A)$ is a discrete subgroup of the vector space $\mathfrak{a} = \text{Lie}(A)$.
- c) There exists $k, l \in \mathbb{N}$ such that $\mathfrak{a}/\Gamma_A \cong \mathbb{R}^k \times T^l$ and in particular the quotient map $\pi_A: \mathfrak{a} \rightarrow \mathfrak{a}/\Gamma_A$ is a smooth map.
- d) The exponential map \exp_A factors through a diffeomorphism $\varphi: \mathfrak{a}/\Gamma_A \rightarrow A$.
- e) There exists a Lie group isomorphism $A \cong \mathbb{R}^k \times T^l$.

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Exercise Sheet no. 5

1. Exercise (4 points).

Let G be a compact Lie group and $H \subset G$ be a closed subgroup. Endow the Lie group with a bi-invariant metric. Show that the closed subgroup H is a totally geodesic submanifold of G , i.e. every geodesic of H is also a geodesic of G . *Bonus: Construct a compact Lie group with a left-invariant metric and a subgroup which is not totally geodesic.*

2. Exercise (4 points).

Let G be a Lie group which acts smoothly on a connected manifold M . Assume that the action is transitive, i.e. for all points $x, y \in M$ there exists a group element $g \in G$ with $x = g \cdot y$. Let $x \in M$ and denote with $G_x := \{g \in G \mid g \cdot x = x\}$ the stabilizer at the point x . Show that:

- The action restricted to the connected component of the identity G_0 is still transitive.
- Let $x \in M$. The connected component G_0 is a normal subgroup in G and we have isomorphisms $\pi_0(G) \cong G/G_0 \cong G_x/(G_x \cap G_0)$.
- There exists an example of a non-trivial group action of a connected Lie group G on a connected manifold such that G_x is not connected. If G_x is connected for a point $x \in M$, then G is already connected.

3. Exercise (4 points).

Let G be a connected Lie group which acts smoothly on a connected manifold M . We denote the group action by $\alpha: G \times M \rightarrow M$ and $\alpha_x := \alpha(\cdot, x): G \rightarrow M$ for a point $x \in M$. Show that:

- If the differential $d_1 \alpha_x: \mathfrak{g} \rightarrow T_x M$ is surjective, then every orbit of the action is an open subset.
- The group action is transitive if and only if the differential $d_1 \alpha_x: \mathfrak{g} \rightarrow T_x M$ is surjective.

4. Exercise (4 points).

We endow $\mathbb{C}\mathbb{P}^n$ with the Fubini-Study metric g_{FS} . In particular the quotient map $\pi: S^{2n+1} \rightarrow \mathbb{C}\mathbb{P}^n, x \mapsto [x]$ is a Riemannian submersion.

- Show that the canonical $U(n+1)$ action on $\mathbb{C}\mathbb{P}^n$ is an isometric action.
- Compute the A -tensor of this Riemannian Submersion.
- Use the O'Neill formula to compute the Riemannian curvature tensor and show

$$K^{(\mathbb{C}\mathbb{P}^n, g_{\text{FS}})}(\text{span}\{d\pi(X), d\pi(Y)\}) = K^{S^{2n+1}}(\text{span}\{X, Y\}) + \frac{3}{4} \|A(X, Y)\|^2$$

for all horizontal vectors X, Y of the Riemannian Submersion. What is the numerical range of the sectional curvature of the Fubini-Study metric?

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Exercise Sheet no. 6

1. Exercise (4 points).

Let $0 < m < n$ and $\text{Gr}(m, n) := \text{Gr}_m(\mathbb{R}^n) = \{V \subset \mathbb{R}^n \mid V \text{ is } m\text{-dimensional subvector space}\}$ be the m -Grassmannian of \mathbb{R}^n , i.e. the collection of all m -dimensional subvector spaces of \mathbb{R}^n .

- a) Show that $\text{GL}(n, \mathbb{R})$ and $\text{O}(n)$ act transitively on $\text{Gr}(m, n)$.
- b) Determine the stabilizers of these actions at the point $\mathbb{R}^m \times \{0\} \in \text{Gr}(m, n)$ and write $\text{Gr}(m, n)$ as a homogeneous space G/H with $G \in \{\text{GL}(n, \mathbb{R}), \text{O}(n)\}$.
- c) Can you give an interpretation of the following homogeneous spaces
 - i) $\text{O}(n)/(\text{O}(m) \times \text{O}(n-m))$
 - ii) $\text{SO}(n)/(\text{SO}(m) \times \text{SO}(n-m))$
 - iii) $\text{GL}(n, \mathbb{R})/(\text{GL}(m, \mathbb{R}) \times \text{GL}(n-m, \mathbb{R}))$
 - iv) $\text{GL}_+(n, \mathbb{R})/(\text{GL}_+(m, \mathbb{R}) \times \text{GL}_+(n-m, \mathbb{R}))$
 - v) $\text{O}(n)/(\text{O}(n_1) \times \dots \times \text{O}(n_k))$ with $n = n_1 + \dots + n_k$.
 - vi) $\text{U}(n)/(\text{U}(n-m) \times \text{U}(m))$

2. Exercise: Symmetric spaces (8 points).

Let (M, g) be a complete, simply-connected Riemannian manifold. In the following you may use – without proof – a theorem by Myers and Steenrod, that says that the isometry group $\text{Isom}(M, g)$ of (M, g) is a Lie group that acts smoothly on M . You also may use that any closed subgroup of a Lie group is a submanifold. Show that the following statements are equivalent:

- i) For any $p \in M$, there exists an isometry $\sigma_p: M \rightarrow M$ with $\sigma_p(p) = p$ and $d_p\sigma_p = -\text{id}_{T_pM}$
- ii) There is a simply-connected Lie group G that acts transitively and isometrically on M and a closed connected subgroup $H \subset G$ such that for some $p \in M$, the map $G/H \rightarrow M, gH \mapsto g \cdot p$ is a well-defined G -equivariant diffeomorphism. Moreover we have Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ such that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h},$$

$$[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p},$$

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$$

holds. Furthermore the canonical isomorphism $T_pM \rightarrow \mathfrak{p}$ induces a scalar product on \mathfrak{p} for which the isotropy representation $\text{Ad} : H \rightarrow \text{GL}(\mathfrak{p})$ is orthogonal (i.e. $\text{Ad}(H) \subset \text{O}(\mathfrak{p})$).

Hints for proving $i) \Rightarrow ii)$: You may take as G the universal covering of $\text{Isom}(M, g)$, and for some fixed $p \in M$, take the isotropy group $H := G_p$. A possible way to prove transitivity is to use the Hopf–Rinow theorem. For topological statements the long exact sequence $\pi_1(H) \rightarrow \pi_1(G) \rightarrow \pi_1(G/H) \rightarrow \pi_0(H) \rightarrow \pi_0(G)$ may be helpful. Show that conjugation $C_{\sigma_p} : G \rightarrow G$ is the identity on $H = G_p$. Establish a relation between the map $G/H \rightarrow G/H$ induced from C_{σ_p} and σ_p itself. Consider $I := \text{Ad}_{\sigma_p}$ which is a Lie algebra automorphism of \mathfrak{g} . Show that $\mathfrak{h} = \ker(I - \text{id})$. Use $I^2 = \text{id}$ in order to define the complement \mathfrak{p} .

Hints for proving $ii) \Rightarrow i)$: You may construct a Lie group automorphism $A : G \rightarrow G$ with $A^2 = \text{id}$ and $\ker(d_{\mathbb{1}} A - \text{id}_{\mathfrak{g}}) = \mathfrak{h}$. Show that the map $G/H \rightarrow G/H$, $gH \mapsto A(g)H$ (why well-defined?) defines an isometry $\sigma_H : G/H \rightarrow G/H$ for a suitable G -invariant Riemannian metric on G/H . This isometry fixes $\mathbb{1}H \cong p$, and defines σ_p . Then define σ_q for $q \in M$, $q \neq p$.

3. Exercise (2 bonus points).

Let (M, g) as in the previous exercise, satisfying i) or equivalently ii). Show that $\nabla R = 0$.

4. Exercise (2 bonus points).

Which of the examples in Exercise 1 have a Riemannian metric such that the universal covering satisfies all the assumptions in Exercise 2.

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Please hand in the exercises until **Tuesday, June 4, 12:00 in the letterbox no. 16.**



Exercise Sheet no. 7

1. Exercise: Inverse Cauchy-Schwarz inequality (4 points).

Let $n \geq 1$ and $\langle \cdot, \cdot \rangle$ be a non-degenerated symmetric bilinearform on a real $n+1$ -dimensional vector space of signature $(n, 1)$. We call a vector $v \in V \setminus \{0\}$ *causal* if $\langle v, v \rangle \leq 0$ holds. Show:

a) We have

$$\langle v, w \rangle^2 \geq \langle v, v \rangle \langle w, w \rangle \quad (1)$$

for all causal $v, w \in V$.

b) The subset of all causal vectors in $V \setminus \{0\}$ has exactly two connected components. If two causal vectors $v, w \in V \setminus \{0\}$ belong to the same component, then $\langle v, w \rangle \leq 0$.

c) If $v, w \in V$ belong to the same component of causal vectors, then the *inverse triangle equality* holds:

$$\sqrt{-\langle v+w, v+w \rangle} \geq \sqrt{-\langle v, v \rangle} + \sqrt{-\langle w, w \rangle}. \quad (2)$$

When does equality hold in (1) or (2)?

2. Exercise: No Hopf-Rinow in Lorentzian geometry (2 points).

We consider the coordinates (x, y) on $\mathbb{R}^2 \setminus \{0\}$ and the Lorentzian metric

$$g(x, y) = \frac{1}{x^2 + y^2} (dx \otimes dy + dy \otimes dx).$$

a) Show that g is a Lorentzian metric.

b) Show that the Lorentzian metric is invariant under the group action

$$\begin{aligned} \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\}) &\rightarrow \mathbb{R}^2 \setminus \{0\} \\ (t, (x, y)) &\mapsto (e^t x, e^t y) \end{aligned}$$

and there is an induced Lorentzian metric \bar{g} on the quotient manifold $M := \mathbb{R}^2 \setminus \{0\} / \mathbb{R}$.

c) Show that the quotient is diffeomorphic to a 2-torus.

d) Determine the geodesic $\gamma: I_{\max} \rightarrow M$ with initial conditions

$$\begin{aligned} \gamma(0) &= [(1, 0)], \\ \dot{\gamma}(0) &= \frac{\partial}{\partial x}|_{\gamma(0)}, \end{aligned}$$

where I_{\max} is the maximal domain of definition.

Hint: prove and use the fact that $\dot{\gamma}(t)$ is lightlike for all t , this reduces the geodesic equation to a simple ODE.

e) Conclude that (M, \bar{g}) is not geodesically complete, i.e. not every geodesic may be extended so that is defined on all of \mathbb{R} . Conclude that there is no analog of the Hopf-Rinow theorem in Lorentzian signature.

3. Exercise (4 points).

Let $m > 0$ and $n \geq 3$. Consider the *Schwarzschild metric*

$$g(t, r, x) = -\left(1 - \frac{2m}{r^{n-2}}\right) dt^2 + \frac{1}{1 - \frac{2m}{r^{n-2}}} dr^2 + r^2 g_{S^{n-1}}(x)$$

on $M = \mathbb{R} \times ((2m)^{\frac{1}{n-2}}, \infty) \times S^{n-1}$. Show that (M, g) solves the *vacuum Einstein equation*, i.e. $G_g + \Lambda g = 0$, where

$$G_g = \text{Ric}^g - \frac{1}{2} \text{scal}^g g$$

is the *Einstein tensor* and $\Lambda \in \mathbb{R}$ is called the *cosmological constant*.

4. Exercise: de Sitter and anti-de-Sitter space (4 points).

Let $\langle \cdot, \cdot \rangle_{n,k}$ be the standard scalar product on $\mathbb{R}^{n,k}$. Consider the *pseudosphere*

$$\mathbb{S}^{n-1,k} := \{x \in \mathbb{R}^{n,k} \mid \langle x, x \rangle_{n,k} = 1\}$$

and the *pseudohyperbolic space*

$$\mathbb{H}^{n-1,k} := \{x \in \mathbb{R}^{n,k} \mid \langle x, x \rangle_{n,k} = -1\}.$$

Show:

- a) $\mathbb{S}^{n-1,k}$ is diffeomorphic to $S^{n-1} \times \mathbb{R}^k$.
- b) $\mathbb{H}^{n-1,k}$ is diffeomorphic to $\mathbb{R}^n \times S^{k-1}$.
- c) Show that $\mathbb{S}^{n-1,1}$ and $\mathbb{H}^{n-1,1}$ with the induced metric from $\mathbb{R}^{n,1}$ are solutions of the vacuum Einstein equation. Determine the constant Λ in both situations.
- d) Show that there exists a closed timelike geodesic in $\mathbb{H}^{n-1,1}$.
- e) (Bonus question) Are there closed timelike geodesics on the universal covering of $\mathbb{H}^{n-1,1}$?

Note: $\mathbb{S}^{n-1,1}$ is called de-Sitter space, and the universal covering of $\mathbb{H}^{n-1,1}$ is called anti-de-Sitter space.

Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024

Prof. Dr. Bernd Ammann, Julian Seipel

Please hand in the exercises until **Tuesday, June 11, 12:00 in the letterbox no. 16.**



Exercise Sheet no. 8

1. Exercise (4 points).

Let V be a $n + 1$ -dimensional real vector space.

- Let g and g' be non-degenerated symmetric bilinearforms of index 1 of V . Assume that for all $v \in V$ we have $g(v, v) = 0$ if and only if $g'(v, v) = 0$. Show that there exists a constant $\lambda \in \mathbb{R}$ with $g' = \lambda g$.
- Show that the constant from a) is positive if $n + 1 \geq 3$.
- Let $A: \mathbb{R}^{n,1} \rightarrow \mathbb{R}^{n,1}$ be an isomorphism of vector spaces for $n + 1 \geq 3$, which maps lightlike vectors to lightlike vectors. Show that there exists a constant $\lambda > 0$ such that $\lambda \cdot A \in O(n, 1)$ holds.
- Show that in the case $n + 1 = 2$ the conclusion of c) does not hold.

2. Exercise (4 points).

Let $\mathbb{R}^{n,1}$ be the Minkowski space. We call a linear map $A: \mathbb{R}^{n,1} \rightarrow \mathbb{R}^{n,1}$ *self-adjoint* if

$$\langle Av, w \rangle_{n,1} = \langle v, Aw \rangle_{n,1}$$

holds for all $v, w \in \mathbb{R}^{n,1}$. We call two linear maps $A, B: \mathbb{R}^{n,1} \rightarrow \mathbb{R}^{n,1}$ *similar* if there exists a $U \in O(n, 1)$ such that $A = UBU^{-1}$ holds. Show that every self-adjoint map $A: \mathbb{R}^{n,1} \rightarrow \mathbb{R}^{n,1}$ is similar to one of the following types: Either it is 1) a diagonal matrix D_{n+1} or

$$2) \begin{pmatrix} D_{n-1} & 0 \\ 0 & a & b \\ & -b & a \end{pmatrix} \quad 3) \begin{pmatrix} D_{n-1} & 0 \\ 0 & \lambda & 0 \\ & \epsilon & \lambda \end{pmatrix} \quad 4) \begin{pmatrix} D_{n-1} & 0 \\ & \lambda & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ & 0 & 1 & \lambda \end{pmatrix},$$

where $a, b, \lambda \in \mathbb{R}$ with $b \neq 0$ and $\epsilon = \pm 1$.

3. Exercise (4 points).

Let $\mathcal{L} := \{x \in \mathbb{R}^{n+1,1} \mid \langle x, x \rangle_{n+1,1} = 0\}$ be the set of all lightlike vectors in Minkowski space. Show:

- The map $\iota: S^n \rightarrow \mathcal{L}, x \mapsto (x, 1)$ is a smooth embedding of smooth manifolds. The scalar multiplication of the ambient Minkowski space gives rise to a bijection $S^n \rightarrow \mathcal{L} / \mathbb{R}_{\neq 0}$. We call the inverse of this map $\pi: \mathcal{L} \rightarrow S^n$. This map is smooth.
- Let $x \in \mathcal{L}$ and $v \in T_x \mathcal{L}$ be spacelike. We consider the positively oriented 2-plane $E_v := \text{span}\{x, v\}$. We take two oriented planes $E = E_v$ and $F = E_w$ for some spacelike vectors $v, w \in T_x \mathcal{L}$. The enclosed angle

$$\angle(E, F) := \arccos \frac{\sqrt{\langle v, w \rangle}}{\sqrt{\langle v, v \rangle} \sqrt{\langle w, w \rangle}}$$

is independent on the choice of v, w and takes values in the interval $[0, \pi]$. This angle can also be computed by $\angle(E, F) = \angle(d_x\pi(v), d_x\pi(w))$, where the latter is angled taken in the standard sphere (S^n, g_{std}) .

- c) Let $A \in O(n+1, 1)$. The restriction of the map A to \mathcal{L} gives rise to a commutative square

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{A} & \mathcal{L} \\ \pi \downarrow & & \downarrow \pi \\ S^n & \xrightarrow{\bar{A}} & S^n \end{array}$$

of diffeomorphisms, where \bar{A} is defined $\pi \circ A \circ \pi^{-1}$. The map \bar{A} is a *conformal diffeomorphism*, i.e. there exists a function $f: S^n \rightarrow \mathbb{R}_{>0}$ such that $\bar{A}^*g_{\text{std}} = f^2g_{\text{std}}$ holds.

4. Exercise: (*Conformal compactification of Minkowski space*) (4 points).

Let $(M, g) = (\mathbb{R}^{n,1}, \langle \cdot, \cdot \rangle_{n,1})$ be the Minkowski space. We choose spherical coordinates $(t, r, x) \in \mathbb{R} \times (0, \infty) \times S^{n-1} \cong M \setminus (\mathbb{R} \times \{0\})$ on the complement of the worldline $\mathbb{R} \times \{0\}$ in Minkowski space. We also consider null coordinates $u = t - r, v = t + r$ and $U = \arctan(u), V = \arctan(v)$ as well as $T = U + V, R = V - U$. Show:

- The map $M \setminus (\mathbb{R} \times \{0\}) \rightarrow \mathbb{R}^2 \times S^{n-2}, (t, rx) \mapsto (T, R, x)$ defines a diffeomorphism onto its image. Give an explicit form of the image $U \subset \mathbb{R}^2 \times S^{n-2}$.
- Determine the pulled-back metric \tilde{g} on U of the conformally transformed metric $\Omega^2g := \frac{4}{(1+u^2)(1+v^2)}g$ on M .
- The space $(M \setminus (\mathbb{R} \times \{0\}), \Omega^2g) \cong (U, \tilde{g})$ isometrically embeds into $(\mathbb{R} \times S^n, -dt^2 + g_{S^n})$ via the map $\mathbb{R}^2 \times S^{n-1} \rightarrow \mathbb{R} \times S^n, (T, R, x) \mapsto (T, \sin(R)x, \cos(R))$ and that this extends to an isometric embedding of (M, Ω^2g) .
- The closure of the image of M in $\mathbb{R} \times S^n$ is compact.
- Determine the following subsets of the closure of the images:

$$\begin{aligned} \mathcal{I}^0 &= \left\{ \lim_{t \rightarrow \infty} \gamma(t) \mid \gamma: \mathbb{R} \rightarrow M \text{ is a spacelike geodesic} \right\} \\ \mathcal{I}^\pm &= \left\{ \lim_{t \rightarrow \infty} \gamma(t) \mid \gamma: \mathbb{R} \rightarrow M \text{ is a future timelike geodesic} \right\} \\ \mathcal{J}^\pm &= \left\{ \lim_{t \rightarrow \infty} \gamma(t) \mid \gamma: \mathbb{R} \rightarrow M \text{ is a future lightlike geodesic} \right\}. \end{aligned}$$

Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024

Prof. Dr. Bernd Ammann, Julian Seipel

Please hand in the exercises until **Tuesday, June 18, 12:00 in the letterbox no. 16.**



Exercise Sheet no. 9

1. Exercise (4 points).

Let (M, g) be a connected, complete Riemannian manifold and $p \in M$. Recall that we defined \mathcal{R}_p as the set of all $q \in M \setminus \{p\}$ with the following properties:

- there is a unique shortest curve $\gamma_q: [0, 1] \rightarrow M$ from p to q . We parametrize γ_q proportional to arclength, thus it is a geodesic with $\|\dot{\gamma}_q(0)\| = \text{dist}(p, q)$.
- the endpoints of γ_q are not conjugate along γ_q .

(Note that these conditions imply, in view of the 4. Exercise, that for any $0 \leq a < b \leq 1$, $\gamma_q(a)$ is not conjugate to $\gamma_q(b)$.) Furthermore we define

$$\mathcal{R}_p^{\text{tan}} := \{\dot{\gamma}_q(0) \mid q \in \mathcal{R}_p\}.$$

We have said, that $\mathcal{R}_p^{\text{tan}}$ is open in $T_p M$ and that \mathcal{R}_p is open in M . Finally, the *cut locus* \mathcal{C}_p at p is given by the complement $M \setminus \mathcal{R}_p$. Determine $\mathcal{R}_p^{\text{tan}}$, \mathcal{R}_p and \mathcal{C}_p for the following cases:

a) $M = \mathbb{R}^2 / \Gamma$ for $\Gamma = \text{span}_{\mathbb{Z}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right)$ and the point $p = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

b) $M = \mathbb{R}P^n$ and the point $p = \mathbb{R}e_1$.

Remark: We consider in both cases the metric on the quotient induces from the standard metrics on \mathbb{R}^2 and S^2 .

2. Exercise (4 points).

Let (M, g) be a complete Riemannian manifold and let γ be a ray, i.e. $\gamma: [0, \infty) \rightarrow M$ is a geodesic parameterised by arc length with $d(\gamma(t), \gamma(0)) = t$ for all $t \in [0, \infty)$.

- a) Show that the function $f(q) := \lim_{t \rightarrow \infty} (d(q, \gamma(t)) - t)$ is well-defined. *Hint: Show the monotony in t and that there is a lower bound.*
- b) Assume that there exists an open subset $U \subset M$ such that $f|_U$ is C^1 . Show that the restriction $f|_U$ is a generalized distance function. *Hint: Imitate the proof in the lecture for the case $f(q) = d(q, A)$ for a subset $A \subset M$.*

Remark: The function f in this exercise is called Busemann function.

3. Exercise (6 points).

Let (M, g) be a connected Riemannian manifold, $p \in M$. Recall that the *injectivity radius* is given by

$$\text{injrads}(p) := \sup\{r > 0 \mid B_r^{T_p M}(0) \subset \mathcal{D}_p \text{ and } \exp|_{B_r^{T_p M}(0)} \text{ is injective}\},$$

where \mathcal{D}_p is the domain of the exponential function. We want to show that the restriction of the exponential map $\exp|_{B_{\text{injrads}(p)}^{T_p M}(0)}$ is an immersion (and thus a diffeomorphism onto its image).

- a) Let $\gamma: [-a, b) \rightarrow M$, $b > 0$ be a geodesic parametrized by arclength with $\gamma(-a) = p$, such that p is conjugate to $\gamma(0)$ along γ . Moreover let $J: [-a, 0] \rightarrow TM$ be a non-vanishing Jacobi field along $\gamma|_{[-a, 0]}$ with $J(-a) = J(0) = 0$. Consider

$$V_\epsilon(t) = \begin{cases} J(t) & \text{for } t \in [-a, -\epsilon] \\ \mathcal{P}_{\frac{1}{2}(t-\epsilon), t} \left(J\left(\frac{1}{2}(t-\epsilon)\right) \right) & \text{for } t \in [-\epsilon, \epsilon] \end{cases}$$

for $\mathcal{P}_{s,t}: T_{\gamma(s)}M \rightarrow T_{\gamma(t)}M$ the parallel transport w.r.t the curve γ . Show that $V_\epsilon(t)$ is a vector field along $\gamma|_{[-a, \epsilon]}$ and that it is the variational vector field of a piecewise C^1 -variation of $\gamma|_{[-a, \epsilon]}$ with fixed endpoints.

- b) Let $\delta > 0$. Show that by renormalizing J and choosing an appropriate ϵ we may achieve that $\|J'(t)\| \geq 1$ and $\|J(t)\| \leq \delta$ holds for $t \in [-\epsilon, 0]$.
- c) Show:

$$\int_{-\epsilon}^{\epsilon} \|V'(t)\|^2 dt = \frac{1}{2} \int_{-\epsilon}^0 \|J'(t)\|^2 dt.$$

- d) Show the estimate:

$$\left| \int_{-\epsilon}^{\epsilon} \langle R(V, \partial_t) \partial_t V \rangle dt - \int_{-\epsilon}^{\epsilon} \langle R(J, \partial_t) \partial_t J \rangle dt \right| \leq 3 \cdot \epsilon \cdot \delta^2 \cdot \sup_M |K(M, g)|.$$

- e) Use the second variation formula for the energy to show that $\gamma|_{[-a, \epsilon]}$ is not a shortest curve between its endpoints.
- f) Show that $a < \text{injrads}(p)$ leads to a contradiction. Conclude that $\exp|_{B_{\text{injrads}(p)}^{T_p M}(0)}$ is an immersion.

Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024

Prof. Dr. Bernd Ammann, Julian Seipel

Please hand in the exercises until **Tuesday, June 25, 12:00 in the letterbox no. 16.**



Exercise Sheet no. 10

1. Exercise (4 points).

- a) Let $F_{\leq}, F_{\geq}: I \rightarrow \mathbb{R}$ be two C^1 functions defined on a common interval I and $\kappa \in \mathbb{R}$. Assume that we have

$$\begin{aligned}F'_{\geq}(t) &\geq \kappa + F_{\geq}(t)^2, \\F'_{\leq}(t) &\leq \kappa + F_{\leq}(t)^2,\end{aligned}$$

for all times $t \in I$. Let $t_0 \in \mathbb{R}$. Show:

$$\frac{d}{dt} \left[(F_{\geq}(t) - F_{\leq}(t)) \cdot \exp \left(- \int_{t_0}^t (F_{\geq}(s) + F_{\leq}(s)) ds \right) \right] \geq 0.$$

Assume moreover that if $F_{\geq}(t_0) \geq F_{\leq}(t_0)$ holds, then we have $F_{\geq}(t) \geq F_{\leq}(t)$ for all $t \geq t_0$.

- b) Let (M^m, g) be a Riemannian manifold with $\text{Ric}^g(X, X) \geq (m-1)g(X, X)$ for all $X \in TM$. Let $f: M \rightarrow \mathbb{R}$ be a generalized distance function and H the mean curvature of a hypersurface $N_p = f^{-1}(f(p))$ for a point $p \in M$. Show:

$$\partial_{\text{grad } f} H \geq 1 + H^2.$$

- c) Let $\gamma: (a, b) \rightarrow M$ be an integral curve of $\text{grad } f$ with $0 \in (a, b)$ and $H(\gamma(0)) = 0$. Show that $a \geq -\frac{\pi}{2}$ and $b \leq \frac{\pi}{2}$. *Hint: Use a) with $\kappa = 1$ and $F_{\leq} = \tan$.*
- d) Assume $I = (-\frac{\pi}{2}, \frac{\pi}{2})$. What can we say about $\lim_{t \rightarrow \pm \frac{\pi}{2}} H(t)$?

2. Exercise (4 points).

Let $\gamma: [0, b) \rightarrow M$ be a geodesic in a Riemannian manifold. Show that

$$\{t \in [0, b) \mid t \text{ is not conjugate to } 0\}$$

is open and dense in $[0, b)$.

3. Exercise (4 points).

- a) Let (M, g) be a Riemannian manifold and $K_0 > 0$. Assume that we have a bound on the sectional curvature $|K(M, g)| \leq K_0$. Let $J: [0, b] \rightarrow TM$ be a non-trivial Jacobi field along a geodesic $\gamma: [0, b] \rightarrow M$ with $J(0) = J(b) = 0$, i.e. $\gamma(0)$ and $\gamma(b)$ are conjugate points. Show that $b \geq \frac{\pi}{\sqrt{K_0}}$ holds. *Hint: Derive the function $h(t) = \|J(t)\|$ twice and write down a differential inequality of second order. Consider the substitution $F(t) = \frac{h'(t)}{h(t)}$ and find a differential inequality for F like in Exercise 1a).*
- b) We equip $\mathbb{C}P^n$ with the Fubini-Study metric g_{FS} , i.e. the metric such that the Hopf fibration $\pi: S^{2n+1} \rightarrow \mathbb{C}P^n$ is a Riemannian submersion. Describe the geodesics in $(\mathbb{C}P^n, g_{FS})$. *Hint: You may use the fact that horizontal lifts $\tilde{\gamma}$ of $\gamma: I \rightarrow \mathbb{C}P^n$ are geodesics, iff γ is a geodesic.*
- c) Compute the cut locus of \mathcal{C}_p for a point $p \in \mathbb{C}P^n$ in $(\mathbb{C}P^n, g_{FS})$.

4. Exercise (4 points).

Consider a Riemannian manifold $(I \times N, ds^2 + g_s)$ for an interval $I \subset \mathbb{R}$ and a family of Riemannian metrics $(g_s)_{s \in I}$, which is smooth in s . Show:

- a) The second fundamental form is given by $\mathbb{I}_{N_s} = -\frac{1}{2}\dot{g}_s$, where $N_s := \{s\} \times N$ and the dot-Notation denotes the derivation w.r.t. the parameter s .
- b) Consider the hyperbolic space

$$\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} \mid -x_{n+1}^2 + \sum_{k=1}^n x_k^2 = -1, x_{n+1} > 0\}$$

and show that the distance function $f(x) = \text{dist}(x, e_{n+1})$ is given by $f(x_1, \dots, x_{n+1}) = \text{arcosh}(x_{n+1})$.

- c) A hypersurface $N_s = f^{-1}(s)$ for $s \in (0, \infty)$ is smoothly diffeomorphic to a sphere.
- d) The second fundamental form of $N_s \subset \mathbb{H}^n$ is given by $\mathbb{I}_{N_s} = -\coth(s) \cdot g_{\text{sphere}}$. In particular, the Weingarten map is given by $S = -\coth(s) \mathbb{1}$.
- e) Conclude that the Riccati inequality for mean curvature (Theorem 2.14 from the lecture) is an equality.

Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024

Prof. Dr. Bernd Ammann, Julian Seipel

Please hand in the exercises until **Tuesday, July 2, 12:00 in the letterbox no. 16.**



Exercise Sheet no. 11

1. Exercise: *Cartan-Hadamard* (4 points).

Let (M, g) be a connected and complete Riemannian manifold. We assume that the sectional curvature is non-positive, i.e. $K(M, g) \leq 0$. Let $p \in M$ and denote with $\exp_p: T_p M \rightarrow M$ the Riemannian exponential map at p .

- Use the Proposition 4.1 from the lecture to show that there are no conjugated points.
- Show that the Riemannian manifold $(N, h) := (T_p M, \exp_p^* g)$ is complete.
- Show that $\exp_p: (T_p M, \exp_p^* g) \rightarrow (M, g)$ is a Riemannian covering.

2. Exercise (4 points).

Let V be a real vector space. We consider $R(t), S(t), S_0 \in \text{End}(V)$ which comply to the Riccati equation

$$\begin{aligned}\dot{S}(t) &= R(t) + S^2(t), \\ S(0) &= S_0\end{aligned}\tag{1}$$

- Assume that $R(t)$ and S_0 is symmetric. Show that the solution $S(t)$ is also symmetric.
- Assume moreover $\frac{\nabla}{dt} R(t) = 0$. Compute a solution of the system (1).
- Consider the case $V = \mathbb{R}^2$ and $R(t) = \begin{pmatrix} t+2 & t-2 \\ t-2 & -t+2 \end{pmatrix}$. Find the solution of the system (1) for $S_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3. Exercise (4 points).

Let (M, g) be a connected Riemannian manifold without any pair of conjugate points (e.g. if $K \leq 0$). For $p \in M$ define

$$C_p := \{\tilde{q} \in M \mid \text{There exists more than one shortest curve between } p \text{ and } \tilde{q}\}.$$

We also assume that $q \in Q_p$ satisfies $d(p, q) = \min\{d(p, \tilde{q}) \mid \tilde{q} \in C_p\}$.

- Let $X_1 \neq X_2 \in T_q M \setminus \{0\}$ and assume that X_1 and X_2 are linearly independent. Show that there exists a $Y \in T_p M$ with $\langle Y, X_1 \rangle < 0$ and $\langle Y, X_2 \rangle > 0$ and $\angle(Y, X_1) = \angle(X_2, Y)$.
- Let $\ell := d(p, q)$ and let $\gamma_i: [0, \ell] \rightarrow M$, $i = 1, 2$ be two different geodesics parametrized by arclength from p to q . Let $X_i := \dot{\gamma}_i(\ell)$. Show $X_1 = -X_2$.
Hint: Assume X_1 and X_2 were linearly independent, choose a vector Y as in a). Then show that there are variations $\gamma_{i,s}$, $s \in (-\epsilon, \epsilon)$ of γ_i , with $\gamma_{1,s}(\ell) = \gamma_{2,s}(\ell)$, and $(d/ds)|_{s=0} \gamma_{1,s}(\ell) = (d/ds)|_{s=0} \gamma_{2,s}(\ell) = Y$. Show that for small $s > 0$, we have $\hat{q}_s := \gamma_{1,s}(\ell) \in C_p$ and $d(p, \hat{q}_s) < d(p, q)$.

Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024

Prof. Dr. Bernd Ammann, Julian Seipel

Please hand in the exercises until **Tuesday, July 9, 12:00 in the letterbox no. 16.**



Exercise Sheet no. 12

1. Exercise (4 points).

Let (M, g) be a complete Riemannian manifold and $(W, h) \subset (M, g)$ a compact Riemannian submanifold. We denote with $\mathcal{N}_M W = (TW)^\perp \subset TM$ the normal bundle of W in M . Moreover we denote with $\exp^{\mathcal{N}_M W} := \exp^M_{|\mathcal{N}_M W}: \mathcal{N}_M W \rightarrow M$ the exponential map associated to the normal bundle.

- a) Show that there exists an $\epsilon > 0$, such that $d_X \exp^{\mathcal{N}_M W}$ is invertible for all $\|X\| < \epsilon$.
- b) Assume moreover that $W^{n-1} \subset M^n$ is a hypersurface of M . Let ν be a unit normal field and S the shape operator of the embedding. Let $c: (-\epsilon, \epsilon) \rightarrow W$ be a smooth curve in W with $c(0) = p \in W$ and $\dot{c}(0) = X \in T_p W$. Show that the Jacobi field $J = \frac{d}{ds} \gamma_s$ associated to the geodesic variation

$$\gamma_s(t) := \exp^{\mathcal{N}_M W}_{c(s)}(t \cdot \nu_{c(s)})$$

satisfies $J(0) = X$ and $\frac{\nabla}{dt} J(0) = -S(X)$.

2. Exercise (4 points).

Let (M, g) be a complete Riemannian manifold and $(W, h) \subset (M, g)$ a compact submanifold. Denote $\mathcal{N}_M W$ and $\exp^{\mathcal{N}_M W}$ as in Exercise 1.

- a) Show that there exists an $\epsilon > 0$ and an open neighbourhood $W \subset U$, such that $\exp^{\mathcal{N}_M W}$ maps the set $\{X \in \mathcal{N}_M W \mid \|X\| < \epsilon\}$ to U .
- b) Assume that $W^{n-1} \subset M^n$ is a hypersurface with unit normal field ν and (M, g) is flat. Show that $d_{\lambda^{-1}\nu_p} \exp^{\mathcal{N}_M W}$ is not invertible, where $p \in W$ is a point and λ is an eigenvalue of the shape operator S_p at p .

3. Exercise (4 points).

Let (M, g) be a Riemannian manifold, whose sectional curvature $K = K(M, g)$ satisfies the bounds:

$$0 < \Lambda_1 \leq K \leq \Lambda_2$$

Let $\gamma: [0, l] \rightarrow M$ be a geodesic parametrized by arclength. We define

$$d := \min\{t > 0 \mid \gamma(t) \text{ is conjugated to } \gamma(0) \text{ along } \gamma|_{[0,t]}\}.$$

Use the Rauch Comparison Theorem to show:

$$\frac{\pi}{\sqrt{\Lambda_2}} \leq d \leq \frac{\pi}{\sqrt{\Lambda_1}}$$

4. Exercise (4 points).

Let (M, g) be a compact Riemannian manifold, which has no conjugated points. Recall that we have defined the function

$$c: SM \rightarrow (0, \infty], \\ X \in S_p M \mapsto \sup\{t > 0 \mid d(p, \exp_p^M(tX)) = t\}.$$

Show that there exists a closed geodesic of length $2 \cdot \min_{X \in SM} c(X)$. *Hint: Use a similar argument as in Exercise 3 of sheet 11.*

Differential Geometry II: Exercises

University of Regensburg, Summer Term 2024

Prof. Dr. Bernd Ammann, Julian Seipel

Please hand in the exercises until **Tuesday, July 16, 12:00 in the letterbox no. 16.**



Exercise Sheet no. 13

1. Exercise (4 points).

Let $n \in \mathbb{N}_{\geq 2}$. Show that there exists a complete Riemannian metric g on \mathbb{R}^n with sectional curvature $K(M, g) > 0$.

2. Exercise (4 points).

Let G be a finite generated group and $\Gamma \subset G$ be a finite generating set. Let dist' be a left-invariant metric on G . We denote with $\text{dist}^{G, \Gamma}$ the word metric with respect to Γ . Show:

a) There exists a constant $C > 0$ such that

$$\text{dist}'(g, h) \leq C \cdot \text{dist}^{G, \Gamma}(g, h)$$

holds for all $g, h \in G$.

b) There exists a constant $C > 0$ such that

$$N_{\text{dist}'}(C \cdot R) := \#\{h \in G \mid \text{dist}'(h, \mathbb{1}_G) \leq C \cdot R\} \geq N_{\Gamma}(R)$$

for all $R \in \mathbb{N}$ holds. The function $N_{\Gamma}(R)$ is given by $\#\{g \in G \mid \text{dist}^{G, \Gamma}(g, \mathbb{1}_G) \leq R\}$.

3. Exercise (4 points).

Consider the upper half-plane model of the hyperbolic space, i.e.

$$\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

with metric $g_x^{\mathbb{H}^n} = \frac{1}{x_n^2} \cdot g_{\text{eucl}}$ for $x \in \mathbb{H}^n$. Show:

a) The map $f: \mathbb{H}^n \rightarrow \mathbb{R}, x = (x_1, \dots, x_n) \mapsto \log(x_n)$ is a generalized distance function.

b) The map f from a) is the Busemann function for all geodesics of the form $\gamma(t) = (x_1, x_2, \dots, x_{n-1}, x_n \cdot e^t)$ for $x = (x_1, \dots, x_n) \in \mathbb{H}^n$.

4. Exercise (4 points).

We consider the 3-dimensional discrete Heisenberg group, i.e.

$$\mathcal{H}_3 := \mathcal{H}_3(\mathbb{Z}) := \left\{ M_{x,y,z} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$$

Denote with $X = M_{1,0,0}, Y = M_{0,1,0}$ and $Z = M_{0,0,1}$ and consider the subset $\Gamma = \{X, Y\}$ of \mathcal{H}_3 . Show:

a) We have $X^a = M_{a,0,0}, Y^b = M_{0,b,0}, Z^c = M_{0,0,c}$ and $X^a Y^b Z^{c-ab} = M_{a,b,c}$. Conclude that Γ is a generating set of \mathcal{H}_3 . *Bonus: Write down the Cayley graph $\text{Cay}^{\mathcal{H}_3, \Gamma}$.*

- b) Let $r, s, t \in \mathbb{Z}$. We have $[X^r, Y^s] = Z^{rs}$. Conclude from that $d^{G,\Gamma}(Z^{rs+t}, \mathbf{1}_{\mathcal{H}_3}) \leq 2|r| + 2|s| + 4|t|$.
- c) Let $k, a, b, c \in \mathbb{Z}$ and choose an $r \in \mathbb{N}$ with $r^2 \leq |k| \leq (r+1)^2$. We write $k = \pm r^2 + t$ with $|t| \leq \frac{|2r+1|}{2}$. Conclude that there exists constants $C, D \in \mathbb{R}$ such that

$$d^{G,\Gamma}(X^a Y^b Z^k, \mathbf{1}_G) \leq C \cdot (|a| + |b| + \sqrt{|k|}) + D.$$

- d) Conclude from the statements above that, for $R > 0$ large enough, we have

$$\left\{ X^a Y^b Z^k \mid |a|, |b|, \sqrt{|k|} \leq \frac{R}{2C} \right\} \subset \bar{B}_\Gamma(R)$$

Show that there is a constant $c > 0$ such that $N_\Gamma(R) \geq cR^4$ for sufficiently large R .

- e) (B2 bonus points) Show that there is a constant $C' > 0$ with the property: if $M_{a,b,k} \in \bar{B}_\Gamma(R)$, then $|a| + |b| \leq R$ and $|k| \leq C' \cdot R^2$. Then show that N_Γ grows polynomially of degree 4.