Seminar on Riemannian Gromov-Hausdorff convergence and weighted differential operators

Summer term 2024

Prof. Bernd Ammann

Monday 14–16, M104

Number of sessions: **12**

Available Dates: 16.4. (Dienstag!), 22.4. (Bernd in Freiburg), 29.4., 6.5., 27.5., 3.6., 10.6., 17.6., 24.6., 1.7., 8.7., 15.7.

Special obstruction:

- April 22: Bernd is in Freiburg
- May 13: Conference in Greifswald
- May 20: Whitsuntide

Talk no. 1: Almost flat manifolds. *16.4.* **(Dienstag!)** JONATHAN GLÖCKLE.
The goal of this talk is the following theorem due to Gromov [24], strengthened a bit by Ruh [45]. We say that a Riemannian metric is ϵ-flat on a closed manifold M, if $(\sup |K^g|)\,\mathrm{diam}(M,g)^2 \leq \epsilon$. A manifold carries an ϵ-flat for any $\epsilon > 0$ if and only if it has a sequence of Riemannian metric g_i, such that (M, g_i) converges to a point in the Gromov-Hausdorff sense.
It is easy to see that any nilmanifold carries an ϵ-flat meric for any ϵ. Gromov's theorem [24] states a converse to this: he shows that there is a constant $\epsilon_n > 0$, $n = \dim M$ such that any ϵ_n-flat closed Riemannian manifold of dimension n is finitely covered by a nilmanifold. [Note in the HIOB seminar it is discussed that if M is of that type then $\pi_1(M)$ (or of balls in $\widetilde{M}$) grows polynomially].
The task of the talk is to present or sketch a proof of this result. It might be good to mainly follow [7]. The latter reference is essentially longer, but one advantage of this reference is according to the authors as follows "... M. Gromov's original publication assumes that the reader is very familiar with several rather different fields and has no difficulties in completing rather unconventional arguments...". Later strengthened this result, essentially by the same method [45].

Supplementary Talk no. 1: Gromov's proof of a Bieberbach theorem for flat space *22.4.* **(Bernd in Freiburg)** N.N..
Prove Bieberbach's theorem following [6]. This talk is tightly related to Talk no. 1, and it might be good to discuss this topic before one goes into the details of the proof in Bieberbach's theorem following [6].

Talk no. 2: Manifolds with polynomial growth in the macroscopic limit. *29.4.* RAPHAEL SCHMIDPETER.
Let M be a closed manifold with $\pi_1(M)$ virtually nilpotent. For an aribtrary

Riemannian metric g on M let $\tilde{g}$ be its publlback to the universal covering $\widetilde{M}$. Discuss that the universal covering $(\widetilde{M}, \epsilon g, p_\epsilon)$ for any choice of base point $p\epsilon \in \widetilde{M}$ converges for $\epsilon \to 0$ in the pointed Gromov-Hausdorff sense to a nilpotent Lie group together with a Carnot-Carathédory Finsler metric, defined via the stable norm. The resulting nilpotent Lie group is graded. A similar result holds for limits of finitely generated almost nilpotent groups with a norm given by a finite set of generators. Reference: [37]; compare this to [25] and [47] (HIOB literature)

Supplementary Talk no. 2: Cheeger–Gromov compactness and the Cheeger finiteness theorem

This talk shall treat the two theorems named above. Main sources are [41], [23]. Additional sources are [40, available?], [42, Chapter 11].

The *Cheeger–Gromov compactness* states the following: Let (M_i, g_i), $i \in \mathbb{N}$ be a sequence of closed Riemannian manifolds of dimension n and we assume that there are constants $D, \Lambda, I_0 \in \mathbb{R}_{>0}$ such that we have for all $i \in \mathbb{N}$:

$$\operatorname{diam}(M_i, g_i) \leq D, \quad |\sec^{g_1}| \leq \Lambda, \quad \operatorname{injrad}(M_i, g_i) \geq I_0.$$

Let $\alpha \in (0, 1)$. Then after passing to a subsequence, there is a closed manifold M_∞ with a Riemannian metric g_∞ of regularity $C^{1,1}$ such that (M_i, g_i) converges in "some" $C^{1,\alpha}$-metric to (M_∞, g_∞). In the literature this theorem is essentially [23, Theorem/Corollary on page 121] or the equivalent theorem [41, Theorem 1.7], and one can even conclude $C^{1,\alpha}$ convergence instead of convergence in the Lipschitz topology [23, abstract]. A slightly more general version is [42, Theorem 11.3.6], called the *Fundamental Theorem of Convergence Theory* in Petersen's book.

The Cheeger finiteness theorem [41, Theorem 1.5], and generalizations are given in [42, Section 11.1]. The original article for Cheeger finiteness is [8].

This subject will be treated in the HIOB seminar and thus we will probably skip this in our seminar.

Talk no. 3: Collapsing sequences of Riemannian manifolds with bounded diameter. *6.5.* Matthias Ludewig.

Now, let (M_i, g_i), $i \in \mathbb{N}$ be a sequence of closed Riemannian manifolds of dimension n and we assume that there are constants D and Λ such that $\operatorname{injrad}(M_i, g_i) \leq D$ and $|\sec^{g_1}| \leq \Lambda$. In contrast to Supplementary Talk no. 2 we assume that a lower bound I_0 on the inhjectivity radius no longer exists. Thus after passing to a subsequence, $\operatorname{diam}(M_i, g_i) \to 0$. This situation is called collapse. After passing to a further subsequence, (M_i, g_i) converges in the Gromov-Hausdorff to a compact metric space (X, d) of lower dimension (e. g., in the Hausdorff sense). In nice situations (X, d) is a again a Riemannian manifold (of lower dimension) and there is a map $F_i \colon M_i \to X$ that turns M_i into a fiber bundle over X, whose fibers are closed manifold that are finitely covered by a nilmanifold. However, in general, X will have singularities, e. g., orbifold singularities, but there are essentially more subtle ones.

The list of literature about this subject is long and not easy too read. The speaker should scroll over the literature and decide what (s)he finds presentable

to the audience. An article by Naber and Tian [35, (pdf)] is maybe good from the modern point of view.
In the talk, it is certainly important to present several examples:

- examples where the fibers are collapsing infranilmanifolds
- examples where orbifold singularities develop in the limit
- examples where complicated types of singularity evolve, see e. g., [35]
- see also the examples [35, Examples 1.1 to 1.4, (pdf)]

Then explain to the audience that away from singularities of dimension $\leq \min\{n-5, \dim X - 3\}$, we have, as expected, an infranilmanifold bundle over an orbifold [35, Theorem 1.1]. A similar statement holds for $\dim X = n-1$ [21]. In total, the goal of the talk should be to raise the understanding of some phenomena and not in going in too much details of the proofs.
Classical literature by Cheeger, Fukaya, Gromov:
[14]; [15], [16]; [18], [20], [19], [21], [22]
More recent literature by Naber and TianCheeger:
[35, (pdf)], [36, (pdf)],

Talk no. 4: $\hat{A}$-genus and collapsing. *27.5.* Roman Schiessl.
We say that connected closed manifold M *admits almost non-negatively curved metrics* if for every $\epsilon > 0$, there is a Riemannian metric g with

$$\sec^g \cdot \operatorname{diam}(M, g)^2 \geq -\epsilon.$$

In the talk we follow an article by Lott [30], and we discuss whether a spin manifold admitting almost non-negatively curved metrics has vanishing $\hat{A}$-genus, i. e., whether it has $\hat{A}(M) = 0$. The article contains several related and interesting statements and Questions.

Talk no. 5: Collapsing for almost Ricci-flat 4-dimensional manifolds. *3.6.* Guadalupe Castillo Solano.
We consider the setting of Talk no. 3 with the additional assumptions $n = 4$ and $\mathrm{Ric}^{g_i} \equiv 0$. We have seen in Talk no. 3 that then X is a Riemannian orbifold, and the fibers of $M_i \to X$ away from the singular points are infranilmanifolds. We explain (and proof if time permits) the main results of [34, (pdf)] which gives a deep unterstanding in this situation.

Talk no. 6: Applications of Gromov-Hausdorff convergence to spectral theory. *10.6.* Julian Seipel.

In order to demonstrate the strength of Gromov–Haussdorff methods we will discuss some applications to spectral theory.
An example is the following result by Vargas

Theorem 6.1 (A. Vargas [48, Theorem 5.4.1]) *Suppose (M, g, χ) is a compact n-dimensional Riemannian spin manifold with $|\sec^g| \leq \Lambda$, $\operatorname{diam}(M, g) \leq D$, $\operatorname{vol}(M, g) \geq V$ and $\operatorname{scal}^g \geq n(n-1)$.*
For every $\delta > 0$ and $\alpha \in (0, 1)$ there is an $\epsilon = \epsilon(n, \Lambda, D, V, \alpha, \delta) > 0$ such that if there are "sufficiently"[1] many Dirac eigenvalues in

$$\left[-\tfrac{n}{2} - \epsilon, \tfrac{n}{2} + \epsilon\right]$$

then the $C^{1,\alpha}$ distance between (M, g) and the standard sphere is at most δ. In particular M is diffeomorphic to S^n.

Sketch the proof so that the audience can see how Gromov–Hausdorff methods are used. Vargas' proof also provides an alternative proof[2] of [3, Theorem 1.6/Corollary 1.7]. These investigations were continued by Saskia Roos (née Voss). Extending Vargas she proved some pinching theorems for Dirac operators [43, Section 6]; please state and prove some of them.
If time admits also discuss the following codimenion-1-collapsing result by S. Roos [44, Corollary 6.1] which is proven by simlar methods and which play a role in Talk no. 9.

Talk no. 7: Gromov–Hausdorff convergence for metric measure spaces.
17.6. N.N..
The goal of this talk is to discuss the extension of Gromov–Hausdorff convergence by including measures. We want to understand, that this concept is more adequate for studying the Laplace–Beltrami operator, i. e., the natural Laplacian on functions.
I reccommend to start with the following example (no reference, as one should be able to do this as an exercise).

Example 7.1 *Let (N, h) be a closed Riemannian manifold, and let $F_i \colon N \to \mathbb{R}{>}\,0$, $i \in \mathbb{N}$, be a sequence of smooth functions. Let $M = N \times S^1 \ni (x, t)$, where $t \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$. Equip M with the metrics*

$$g_i|_{(x,t)} := h|_x + F_i(x)^2\, dt^2.$$

(1) *If $\|F_i\|_{L^\infty} \to 0$, then (M, g_i) converges in the Gromov-Hausdorff sense to (N, h).*

(2) *Let $\phi_i := \log F_i$. If there is a constant $C > 0$, such that $\|\nabla\phi_i\|_{L^\infty} < C$ and $\|\nabla^2\phi_i\|_{L^\infty} < C$, then the curvature of (M, g_i) is uniformly bounded. (The audience may believe this, or you may cite formulas from Besse's book on Einstein manifolds, or Section 2.5 of my partial script "Diffgeo II, summer term 2021" https://ammann.app.uni-regensburg.de/lehre/2021s_diffgeo2*

(3) *Make a Fourier decomposition along S^1. Eigenvalues λ for non-zero modes satisfy $\lambda \geq \|F_i\|_{L^\infty}^{-2}$.*

[1] See Vargas' theorem for details
[2] There is no reference for this alternative proof, it is obvious how to get this modification

(4) *Suppose that* $\phi_i \to \phi_\infty$ *in* $C^2(N)$. *Discuss that* $\operatorname{spec}(\Delta^{M,g_i})$ *does not converge to* $\operatorname{spec}(\Delta^{N,h})$, *but to the spectrum of the weighted Laplacian for* $(N, h, e^{\phi_\infty} \operatorname{dvol}^h)$, *where one defines as usual* $\Delta := d^*d$, *but where the adjoint* d^* *is taken with respect to the* L^2*-norms associated to the measure* $e^{\phi_\infty} \operatorname{dvol}^h$.

As a conclusion $\operatorname{spec}(\Delta)$ *is not continuous with respect to Gromov–Hausdorff convergence.*

Now introduce Gromov–Hausdorff convergence with measure as in [27, Chapter $3\frac{1}{2}$, Sections A.$3\frac{1}{2}$.1–B.$3\frac{1}{2}$.3]. You may discuss some examples at the end of these pages, and add some facts that you like in the following sections. (Note that this part is not contained in the original, French version of Gromov's book [26].)
Then discuss the main results [17] which shows the continuity of the spectrum of the Laplace–Beltrami operator with respect to measured Gromov–Hausdorff convergence. If time permits, explain some ideas of the proof.

Talk no. 8: Analysis on metric measure spaces. *24.6.* N.N..
Metric spaces with "nice" measures often arise as measured Gromov–Hausdorff limits. Amazingly much of analysis can be adapted to such metric measure spaces. As an example we study work by Cheeger: [9].
Such ideas give rise to a whole research domain which we cannot treat here; examples are [29], [28], [46] or many articles connected to optimal transport.

Talk no. 9: Spectra of other natural differential operators in Gromov–Hausdorff limits. *1.7.* N.N..
The goal of this talk is to discuss to which extent similar phenomena hold for other natural geometric differential operators.
We start by reconsidering a generalization of Example 7.1 for the Dirac operator. This is discussed in [1], see also [2, Kapitel 7]. Amazingly, in contrast to the Laplace-Beltrami operator there is no effect of the limit measure $e^{\phi_\infty} \operatorname{dvol}^h$. Explain in detail how to identify spinors on the base with zero mode spinors on the circle bundle. This can be generalized to bundles of k-dimensional toruses. Now assume that M is a warped product, similar to Example 7.1, but with a k-torus. Formulas for spinors on the k-torus bundle give formulas for spinors on N in a weighted sense. E.g. for any $k \in \mathbb{N}_{>0}$ we get a weighted Schrödinger–Lichnerowicz formula, a weighted positive mass theorem, etc. Such formula will play an important role in Talk no. 10.
One then might ask whether the Dirac spectrum may possibly be continuous with respect to Gromov–Hausdorff convergence. Howver, a counterexample is already given by Heisenberg nilmanifolds: the curvature term $\mathcal{F}$ in [44, Corollary 6.1] survives in the limit (thus in order to get continuity for Dirac eigenvalues one should pass to Gromov–Hausdorff convverges with additional data, that includes the curvature of the collapsing infranilmanifolds – which is not so easy to formalize).
A more general approach to the behavior of collapsing of natural geometric differential operators was followed by J. Lott, which led to some articles about the

Dirac spectrum [31, (pdf)] and the spectrum of the Hodge Laplacian [33, (pdf)], [32, (pdf)]. If time permits,also explain about this pproach. Some photocopied slides of a previous talk by Lott (ask Bernd) might help.

Talk no. 10: Weighted Dirac operators. *8.7.* N.N.
The topic of this talks are weighted Dirac operators. These are similarly defined as the weighted Dirac operators in Talk no. 9, howver with adifferent constant. Such operators were already mentioned in Perelman's breakthrough article towards geometrization in dimension 3 [39, Remark 1.3, (pdf)], but its claimed connection to Perelman's $\mathcal{F}$-functional — also called λ-entropy — remained mysterious.
J. Baldauf and T. Ozuch recently published two articles [4, (pdf)] [5, (pdf)] in which the weighted Dirac operator is systematically studied and it was amazing how far classical results about spinors, scalar curvature and the ADM mass found theor weighted counterparts. To some extension this is not a conincidence but a consequence of the weighted Dirac operators in Talk no. 9. Let $(M, g, e^{-f}\,\mathrm{dvol}^g)$ be a weighted Riemannian spin manifold, $f\colon M \to \mathbb{R}_{>0}$. We equip $P = M \times T^k$ with a warped product metric $G := g + e^{-2f/k} g_{\text{flat}}$, where g_{flat} is a fixed flat metric on T^k.
Consider a classical spinorial relations, e.g. the Schrödinger-Lichnerowicz formula on P. Restriction to T^k-invariant spinors and identifying these invariant spinors with spinors on M as in Talk no. 9 provides a spinorial equation on M. This is *almost* the Lichnerowicz formula derived in [4], some constant (depending on k) are different. Howver, linearly combining the equations for two values of $k \in \mathbb{N}4$, one obtains all *equalities* [4] — however, a priori, not the ineqqualities. This approach will save many calculations, however it was not yet worked out so far.
The article [4] shows many relations between weigthed ADM mass and the Ricci flow. In particular it shows that the weigted ADM mass is better adapted to study the Ricci flow on asymptotically euclidean manifolds. While the standard ADM mass remains invariant under Ricci flow, it is shown that the weighted ADM mass is monotonically decreasing under Ricci flow and it measures the evolution towards flat space. In fact it coincides (up to sign) with the ALE-entropy from Deruelle and Ozuch. One goal of the talk is to explain these relations from [4].
Be aware that the claimed "Ricci identity" is incorrect (the right hand side is tensorial, while the left hand side is not) – this gap, however was taken over from previous literature in spin geometry.
The other part of the talk shall summarize [5, (pdf)]. In equation (0,3) a functional

$$\mathcal{E}_g(\psi, f) = \int_M \Big(4|\nabla\psi|^2 + \mathrm{scal}^f \cdot (|\psi|^2 - 1)\Big)\, e^{-f}\,\mathrm{dvol}^g$$

is defined, depeding on a Riemannian metric g, a weight function f and a spinor ψ. By taking a minimum over ψ and a maximum over f one obtains a functional on the space of Riemannian metrics, denote $\kappa(g)$. Explain that Ricci flow is the $L^2(e^{-f}\,\mathrm{dvol}^g)$-gradient flow of κ, see [5, Theorem 0.8].

Talk no. 11: Sequences of Riemannian manifolds with Ricci curvature bounded from below. *15.7.* N.N..

Many phenomena we have encountered in this seminar still hold under weaker curvature conditions. Strong results exists if one replaces $|\sec| \leq C$ by $\sec \geq -C$, $|\mathrm{ric}| \leq C$ or $\mathrm{ric} \geq -C$. This talk summarizes the situation for the assumption $\mathrm{ric} \geq -C$, i. e., if the Ricci curvature is bounded from below. References are [10], [11], [12], [13].

Further topics and literature

Further article on GH convergence [38]

Seminar-Homepage

https://ammann.app.uni-regensburg.de/Gromov-Hausdorff

Literatur

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