Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, February 6**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 14

1. Exercise (4 points).
Let $\varphi\colon (M, g) \to (N, h)$ be a smooth map between connected manifolds and $g = \varphi^* h$ is the pullback of the metric h.

a) If φ is a covering map, then show that (M, g) is complete iff (N, h) is complete.

b) Assume that φ is a local diffeomorphism and an isometry. Show that if (M, g) is complete, then the map φ is a covering map.

2. Exercise (4 points).
Let $(M^{n\geq 2}, g)$ be connected, complete Riemannian manifold with constant sectional curvature. Assume moreover that M is simply-connected. Show

$$(M, g) \text{ is isometric to } \begin{cases} \mathbb{H}^n & \text{if } K = -1, \\ \mathbb{R}^n & \text{if } K = 0, \\ \mathbb{S}^n & \text{if } K = 1. \end{cases}$$

3. Exercise (4 points).
Let $\varphi\colon (M, g) \to (N, h)$ be a surjective submersion between connected complete Riemannian manifolds. We call φ a *Riemannian submersion* if the map $d_p\varphi$ induces an isomorphism $H_pM := (\ker(d_p\varphi))^\perp \to T_{\varphi(p)}N$ for each $p \in M$. We call $HM := \bigcup_{p\in M} H_pM \subset TM$ the horizontal subbundle and its elements *horizontal*.

a) Let $\gamma : I \to N$ be a smooth curve, I some interval. Show that there exists a horizontal lift $\tilde{\gamma} : I \to M$, i.e. a curve $\tilde{\gamma}$ satisfying $\dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}(t)}M$ and $\varphi \circ \tilde{\gamma} = \gamma$. Also show for any curve $\tau : [a, b] \to N$ that $\mathcal{L}(\varphi \circ \tau) \leq \mathcal{L}(\tau)$.

b) Show: if γ is a geodesic, then its horizontal lift $\tilde{\gamma}$ is also a geodesic. *Hint: use the fact that γ locally minimizes length to show that $\tilde{\gamma}$ also minimizes length locally.*

c) Show: if a horizontal curve $\tau : I \to M$ is geodesic, then $\varphi \circ \tau : I \to N$ is also a geodesic.

d) Let γ be a geodesic in M. Show that if $\dot{\gamma}(0)$ lies in $H_{\gamma(t)}M$ then we have $\dot{\gamma}(t) \in HM$ for all $t \in I$.

4. Exercise (4 points).
Let (M^n, g) be a Riemannian manifold. We assume that (M, g) is *locally symmetric*, i.e. $\nabla R = 0$ holds. In this exercise we want to show that this condition is equivalent to the existence of a local isometry $\sigma_p : U \to \sigma(U)$ with $\sigma(p) = p$ and $d_p\sigma = -\mathrm{id}_{T_pM}$, defined on open neighbourhood $U \subset M$ of p.

a) Let $\epsilon > 0$ small enough such that the exponential function is a diffeomorphism onto its image, i.e. $\exp_p: B_\epsilon(0) \xrightarrow{\simeq} \exp_p(B_\epsilon(0)) = B_\epsilon(p)$. We define the map

$$\begin{aligned} \sigma_p: B_\epsilon(p) &\to B_\epsilon(p) \\ \gamma(t) &\mapsto \gamma(-t), \end{aligned}$$

where we use that each point in $B_\epsilon(p)$ can be represented by a geodesic emanating from p. Show that $\sigma_p = \exp_p \circ(-\operatorname{id}_{T_pM}) \circ \exp_p^{-1}$ holds.

b) Let $v \in B_\epsilon(0)$ and $q = \exp_p(v)$. Moreover let $\gamma(t) = \exp_p(tv)$ and $\bar{\gamma}(t) = \gamma(-t)$ be curves in M. We consider the map

$$\begin{aligned} F_t : T_{\gamma(t)}M &\to T_{\bar{\gamma}(t)}M \\ w &\mapsto \mathcal{P}^{\bar{\gamma}}_{0,t} \circ (-\operatorname{id}_{T_pM}) \circ \mathcal{P}^{\gamma}_{t,0}(w), \end{aligned}$$

where $\mathcal{P}^c_{a,b}: T_{c(a)}M \to T_{c(b)}M$ denotes the parallel transport along the curve $c: I \to M$ with $a, b \in I$. Show that for each Jacobi field $J(t)$ along γ, the field $\bar{J}(t) = F_t(J(t))$ is a Jacobi field along $\bar{\gamma}$. Conclude from the previous statement that the map $\sigma_p: B_\epsilon(p) \to B_\epsilon(p)$ is an isometry.

c) Let $\gamma: (-\epsilon, \epsilon) \to M$ be a geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Moreover assume that (M, g) is not nessecarily locally symmetric and all the maps σ_p from part a) are isometries. Show for a parallel frame $(e_1(t), \dots, e_n(t))$ along γ we have

$$g_{\gamma(t)}(R(e_i(t), e_j(t)e_k(t)), e_l(t)) = g_{\gamma(-t)}(R(e_i(-t), e_j(-t)e_k(-t)), e_l(-t))$$

and conclude that (M, g) is locally symmetric.