Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, January 30**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 13

1. Exercise (4 points).
Let M be a compact surface (without boundary) in $\mathbb{R}^3$. Let $\overline{B}_r(0)$ be the closed ball of radius r around 0 in $\mathbb{R}^3$, and let $S_r(0) = \partial\overline{B}_r(0)$ be its boundary.

a) Show that the infimum $R := \inf\{r > 0 \,|\, M \subset \overline{B}_r(0)\} > 0$ is attained, and conclude that $M \cap S_R(0)$ is not empty.

b) Show that T_pM is the orthogonal complement of p for any $p \in M \cap S_R(0)$. Show for any such $p \in M$ that the symmetric bilinear form

$$T_pM \times T_pM \to \mathbb{R}, \quad (X, Y) \mapsto \left\langle \tfrac{1}{R}p, \vec{\mathrm{II}}(X, Y)\right\rangle$$

is negative definit.

c) Are there compact minimal surfaces M in $\mathbb{R}^3$? Justify your answer.

2. Exercise (4 points).
Let (M, g) be a connected, non-compact, geodesically complete Riemannian manifold and $p \in M$ be a point. You may use the facts that under these conditions (M, d) is a complete metric space and that for any $p, q \in M$ there is a shortest curve from p to q.

a) Show the existence of a sequence points $\{p_i\}_{i\in\mathbb{N}}$ in M with $d(p, p_i) \to \infty$ for $i \to \infty$.

b) Conclude the existence of a geodesic ray[1] $\gamma\colon [0, \infty) \to M$ with $\gamma(0) = p$.
Hint: Consider a length minimizing geodesic $\gamma_i\colon [0, l_i] \to M$ with $\gamma_i(0) = p$ and $\gamma_i(l_i) = p_i$. Use the fact that $\|\dot\gamma_i(0)\| = 1$ to conclude that there exists convergent subsequence $\dot\gamma_{i_j}(0) \to X \in T_pM$. Consider then $\gamma(t) = \exp_p(tX)$ and show $d(p, \gamma(t)) = t$.

3. Exercise (4 points).
Let (M, g) be a connected, geodesically complete Riemannian manifold and $N \subset M$ be a closed submanifold.[2] We fix a point $q \in M \smallsetminus N$. We denote by $d(x, N) := \inf\{d(x, y) \mid y \in N\}$ the minimal distance from x to the submanifold N.

a) Show that there exists a point $p \in N$ with $d(q, p) = d(q, N)$. Do we need the assumption that N is closed?

b) Show the existence of a geodesic γ, which connects p and q with length given by $\mathcal{L}(\gamma) = d(q, p)$.

c) Conclude with the first variation of the energy that the curve γ hits N in an orthogonal way.

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[1] A geodesic ray $\gamma\colon [0, \infty) \to M$ is a geodesic such that for all compact subsets $K \subset M$ there exists a time $T > 0$ such that $\gamma(T) \notin K$ holds.

[2] You may use the facts that under these conditions (M, d) is a complete metric space and that for any $p, q \in M$ there is a shortest curve from p to q.

4. Exercise (4 points).
Let M be a smooth manifold and G be a group equipped with the discrete topology. Moreover we have a continuous group action

$$\begin{aligned} R\colon M \times G &\to M \\ (p, g) &\mapsto R(p, g), \end{aligned}$$

i.e. R satisfies $R(p, gh) = R(R(p, g), h)$ for all $p \in M$ and $g, h \in G$. We denote by $p \cdot G := \{R(p, g) \mid g \in G\}$ the orbit of p along the group action and we denote by $M/G := \{p \cdot G \mid p \in M\}$ the *quotient space* of the group action. The *canonical projection* $\pi\colon M \to M/G, p \mapsto \pi(p) = p \cdot G$ induces a topology on the quotient M/G, i.e. a subset $U \subset M/G$ is open iff $\pi^{-1}(U) \subset M$ is open.

a) Show that the right multiplication maps $R_g\colon M \to M, p \mapsto R(p, g)$ is a homeomorphism for any $g \in G$. Are these maps also diffeomorphisms?

Now we assume that the group action R is free and properly discontinuous. Here we refer to an action R as free if for any $g \in G \setminus \{e\}$ the right multiplication maps R_g has no fixed point. An action R is properly discontinuous if for all points $p, q \in M$ there exist open neighbourhoods U_p, V_q of p respectively q such that $R_g(U_p) \cap V_q = \varnothing$ holds for all $g \in G$ with the condition $R(p, g) \neq q$.

b) Show that the quotient space M/G is Hausdorff.

c) Show that the canonical projection $\pi\colon M \to M/G$ is a covering map, i.e. for all points $P \in M/G$ there exists an open neighbourhood U of P and a homeomorphism $\Phi_U\colon \pi^{-1}(U) \to U \times G$ such that $\Phi \circ \mathrm{pr}_1 = \pi$ holds.

d) (Bonus part) Assume additionally that R_g is smooth for any $g \in G$. Show then that the quotient space M/G is a smooth manifold and the canonical projection is a local diffeomorphism.