Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, January 23**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 12

1. Exercise (4 points).
Let (M^2, g) be a two-dimensional Riemannian submanifold of $\mathbb{R}^3$. We call M a *minimal surface* if the mean curvature of M in $\mathbb{R}^3$ vanishes.

a) Show that a minimal surface has non-positive sectional curvature, and if the sectional curvature is 0 in $p \in M$, then the fundamental form vanishes in p.

b) Consider the *catenoid*

$$\Phi_1 \colon \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x, y) \mapsto \begin{pmatrix} \alpha \cosh(x) \cos(y) \\ \alpha \cosh(x) \sin(y) \\ \sinh(x) \end{pmatrix}$$

and the *helicoid*

$$\Phi_2 \colon \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x, y) \mapsto \begin{pmatrix} x \cos(y) \\ x \sin(y) \\ \beta y \end{pmatrix}$$

with constants $\alpha, \beta \in \mathbb{R}$. Compute the induced metrics g_1, g_2 on $\mathbb{R}^2$ and the Weingarten maps. Show that the catenoid and the helicoid are minimal surfaces in $\mathbb{R}^3$.

c) Compute the sectional curvatures of both surfaces. Does there exists an isometry $\phi \colon (\mathbb{R}^2, g_1) \to (\mathbb{R}^2, g_2)$?

d) Show that there does not exists an isometry $\bar{\phi} \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\bar{\phi}(\mathrm{image}(\Phi_1)) = \mathrm{image}(\Phi_2)$ holds.

2. Exercise (4 points).
Let (M^n, g) be a Riemannian manifold with non-positive sectional curvature, i.e. $K \leq 0$. We denote by J a Jacobi field along a geodesic c of (M, g).

a) Show that $g(J, \frac{\nabla^2}{dt^2} J)$ is a non-negative function.

b) Show that $\frac{d^2}{dt^2}(g(J, J))$ is a non-negative function.

c) Conclude from the previous statements that the Jacobi field vanishes identically or has at most one point where it vanishes.

3. Exercise (4 points).
Let (M,g) be a semi-Riemannian manifold and J be a Jacobi field along a geodesic $c: I = [a,b] \to M$. Show that there exists a geodesic variation $c_\bullet: (-\epsilon,\epsilon) \times I \to M$ of c such that $J = \frac{d}{ds}|_{s=0} c_s$ holds.
Hint: For some $t_0 \in [a,b]$ choose a curve $\gamma : (-\epsilon,\epsilon) \to M$ with $\gamma(0) = c(t_0)$ and $\dot\gamma(0) = J(t_0)$. Find a vector field X along γ such that $(s,t) \mapsto c_s(t) = \exp_{\gamma(s)}(tX(s))$ is a suitable geodesic variation.

4. Exercise (4 points).
Let (M,g) be a semi-Riemannian manifold.

a) Recall that we denote the parallel transport along a curve γ by $\mathcal{P}_\gamma$. Let $F: \mathbb{R}^2 \to M$ be a smooth map and denote by γ_t the curve in M which is given by

$$\gamma_t(s) = \begin{cases} F(4st, 0) & s \in [0, \frac{1}{4}] \\ F(t, t(4s-1)) & s \in [\frac{1}{4}, \frac{1}{2}] \\ F(t(3-4s), t) & s \in [\frac{1}{2}, \frac{3}{4}] \\ F(0, t(4-4s)) & s \in [\frac{3}{4}, 1], \end{cases}$$

i.e. the piecewise smooth curve which gives the image of the closed polygonal chain with corner points $(0,0), (t,0), (t,t)$ and $(0,t)$. Show that

$$\lim_{t\to 0} \frac{\mathcal{P}_{\gamma_t} v - v}{t^2} = R\left(\frac{\partial F}{\partial x_2}(0), \frac{\partial F}{\partial x_2}(0)\right) v$$

holds for all $v \in T_{F(0,0)}M$.
Hint: Use the following statement from the lecture (Lemma V.4.2): Let $\alpha: \mathbb{R}^2 \to M$ be a smooth map and X a vector field along α such that $\frac{\nabla}{\partial x} X = \frac{\nabla}{\partial y} X$ holds, then we have

$$\frac{\nabla}{\partial x}\frac{\nabla}{\partial x} X - \frac{\nabla}{\partial y}\frac{\nabla}{\partial y} X = R\left(\frac{\partial \alpha}{\partial x}, \frac{\partial \alpha}{\partial y}\right) X.$$

b) If (M,g) is flat, then for every point $p \in M$ and vector $v \in T_pM$, there exists an open neighbourhood of p given by $U \subset M$ and a section $X: U \to TM$ of the tangent bundle TM, which is parallel, i.e. $\nabla X = 0$ on U, and satisfies $X_p = v$. Construct a counterexample in the non-flat case for the previous statement.