Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, January 16**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 11

1. Exercise (4 points).
Let (M,g) be a semi-Riemannian manifold and $N \subset M$ be an open subset. Assume that N is geodesically complete[1] and M is connected. Show that $N = M$ holds. *Hint: Consider a point in the boundary* $\bar{N} \setminus N$*.*

2. Exercise (4 points).
Let $N \subset M$ be a semi-Riemannian submanifold of the semi-Riemannian manifold (M,g). We say that N is *totally geodesic* if the second fundamental form $\vec{\mathrm{II}} \equiv 0$ vanishes.

a) Show that N is totally geodesic iff every geodesic of N is also a geodesic of M.

b) Assume now that N is geodesically complete. Show that N is totally geodesic iff every geodesic $\gamma: I \to M$, of M with $\dot{\gamma}(0) \in TN$ is contained in N.

c) Do we need the assumption of geodesic completeness in part b) to conclude the statement?

3. Exercise (4 points).
Let $(\bar{M},\bar{g})$ be a semi-Riemannian manifold and M be submanifold of dimension $n = \dim(M) = \dim(\bar{M}) - 1$. Assume that there exists a map into the normal bundle $\nu: M \to \mathcal{N}M$, such that $g(\nu,\nu) = \epsilon \in \{-1,+1\}$ holds. Denote by g the induced Riemannian metric on M.

a) Show that there exists a unique bundle map $W \in \Gamma(\mathrm{End}(TM))$ with the property

$$g(W(X),Y) = \bar{g}(\vec{\mathrm{II}}(X,Y),\nu)$$

for all $X, Y \in T_pM$ and $p \in M$. In particular, the endomorphism $W|_p: T_pM \to T_pM$ is self-adjoint. We call W the *Weingarten map* of the embedding $(M,g) \hookrightarrow (\bar{M},\bar{g})$.

b) Show that $W(X) = -\bar{\nabla}_X \nu$ holds for all $X \in TM$.

c) Assume that $\bar{M}$ is Riemannian and $n = \dim(M) \geq 3$. Moreover the metric on $\bar{M}$ is assumed to be flat, i.e. $\bar{R} \equiv 0$. Show that for any point $p \in M$ there is a plane $E \subset T_pM$ with $K(E) \geq 0$. *Hint: Consider planes* $E = \mathrm{span}(\xi_i, \xi_j)$ *which are spanned by an orthonormal basis* $\xi_1, \ldots, \xi_n$ *of eigenvectors of* W *and use the Gauß formula.*

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[1] A semi-Riemannian manifold N is *geodesically complete* if the exponential map is defined on the full tangent bundle TN

4. Exercise (4 points).
Let $(\bar{M}, \bar{g})$ be a flat semi-Riemannian manifold and M be a semi-Riemannian submanifold of $\bar{M}$ with dimension m and induced metric g. Let $(b_1, \dots, b_m)$ be a generalized orthonormal basis of T_pM with the condition $g(b_i, b_j) = \delta_{ij}\,\varepsilon_i$, $\varepsilon_i \in \{-1, 1\}$. We define the *mean curvature vector field* by $\vec{H}_p := \sum_{i=1}^m \varepsilon_i\, \vec{\mathrm{I\!I}}(b_i, b_i)$.

a) Show that $\vec{H}_p$ is well-defined.

b) Show that

$$\mathrm{Ric}(X, Y) = \bar{g}(\vec{H}_p, \vec{\mathrm{I\!I}}(X, Y)) - \sum_{i=1}^{m} \varepsilon_i\, \bar{g}(\vec{\mathrm{I\!I}}(b_i, X), \vec{\mathrm{I\!I}}(b_i, Y)).$$

holds for all $X, Y \in T_pM$

c) Let M be of dimension $m - 1$ and assume that there exists a map into the normal bundle $\nu\colon M \to \mathcal{N}M$, such that $g(\nu, \nu) = \epsilon \in \{\pm 1\}$ holds with associated Weingarten map W (defined in Exercise 3). Show that:

$$\bar{g}(\nu, \nu) \cdot \mathrm{scal} = (\mathrm{Tr}\, W)^2 - \mathrm{Tr}(W^2).$$