Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl
Please hand in the exercises until **Tuesday, January 9**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 10

1. Exercise: *Polar normal coordinates* (4 points).
Let (M^2, g) be a 2-dimensional Riemannian manifold. Let $p \in M$ be a point and choose an $\epsilon > 0$ such that the exponential map $\exp_p : B_\epsilon(0) \to \exp_p(B_\epsilon(0))$ is a diffeomorphism. Denote by $x = (x^1, x^2)$ the normal coordinates at p and consider the induced *Polar normal coordinates* (r, φ) via the identification $T_pM \cong \mathbb{R}^2$ with euclidean space.

a) Show that we have the following identification of the induced coordinate vector fields:

$$\begin{aligned}\frac{\partial}{\partial r} &= \cos(\varphi)\frac{\partial}{\partial x^1} + \sin(\varphi)\frac{\partial}{\partial x^2}\\ \frac{\partial}{\partial \varphi} &= -r\sin(\varphi)\frac{\partial}{\partial x^1} + r\cos(\varphi)\frac{\partial}{\partial x^2}\end{aligned}$$

b) Determine the coefficients of the metric in Polar normal coordinates $g_{rr}, g_{r\varphi}, g_{\varphi\varphi}$ in terms of the metric g_{ij} with respect to normal coordinates.

c) Let (E_1, E_2) be an orthonormal basis of (T_pM, g_p). Consider the closed curve $\gamma_r(t) = \exp_p(r\cos(t)E_1 + r\sin(t)E_2)$ on M for $t \in [0, 2\pi]$ and a radius $r < \epsilon$. Show that the sectional curvature K_p of (M, g) at p can be computed as follows

$$K_p = \frac{3}{\pi}\lim_{r\to 0}\frac{2\pi r - \mathcal{L}[\gamma_r]}{r^3},$$

where $\mathcal{L}[\gamma_r]$ is the length of the curve γ_r. Can you give a heuristic explanation of this formula? *Hint: Use the Taylor expansion of the metric in normal coordinates and express it then in Polar normal coordinates.*

2. Exercise: *Bianchi identities* (4 points).
Let $\alpha \in \Omega^1(M)$ be a 1-form and $\beta \in \Omega^2(M)$ be a 2-form on a Riemannian manifold (M, g). Let $X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$ be vector fields on M. Recall the expressions of the Cartan differential:

$$\begin{aligned}d\alpha(X_1, X_2) &= X_1(\alpha(X_2)) - X_2(\alpha(X_1)) - \alpha([X_1, X_2]),\\ d\beta(X_1, X_2, X_3) &= \sum_\sigma X_{\sigma(1)}(\beta(X_{\sigma(2)}, X_{\sigma(3)})) - \beta([X_{\sigma(1)}, X_{\sigma(2)}], X_{\sigma(3)}),\end{aligned}$$

where the sum in the second formula runs over all cyclic permutations of the set $\{1, 2, 3\}$.

a) Show:

$$d\alpha(X_1, X_2) = (\nabla_{X_1}\alpha)(X_2) - (\nabla_{X_2}\alpha)(X_1)$$

b) Use $dd\alpha = 0$ to deduce the first Bianchi identity:

$$R(X_1, X_2)X_3 + R(X_2, X_3)X_1 + R(X_3, X_1)X_2 = 0$$

c) Let $X \in \mathfrak{X}(M)$ be a fixed vector field. Define $\tilde{\alpha}(X_1) = \alpha(\nabla_{X_1} X)$ and deduce, by using $dd\tilde{\alpha} = 0$, the second Bianchi identity:

$$(\nabla_{X_1} R)(X_2, X_3) + (\nabla_{X_2} R)(X_3, X_1) + (\nabla_{X_3} R)(X_1, X_2) = 0$$

3. Exercise (4 points).
Let (M^n, g) be a Riemannian manifold. Denote by R the Riemannian curvature tensor as a $(1,3)$-tensor. Let $X, Y, Z, U, W \in \mathfrak{X}(M)$ be vector fields on M, then define

$$R^{(0,4)}(X, Y, Z, W) = g(R(X, Y)Z, W)$$
$$g(R^{\Lambda^2}(X \wedge Y), Z \wedge W) = R(X, Y, Z, W)$$

the associated $(0,4)$-tensor and curvature endomorphism.

a) Let $\{e_i\}_i \subset T_pM$ be an orthonormal basis of g. Show that by

$$g_p(e_i \wedge e_j, e_k \wedge e_l) = \delta_{ik}\delta_{jl}$$

for $i < j$ and $k < l$ we obtain a non-degenerated bilinearform on T_pM, which depends smoothly on p.

b) Show that $R^{(0,4)}$ is a well-defined $(0,4)$-tensor on M and R^{Λ^2} is a well-defined map $\Lambda^2 T_pM \to \Lambda^2 T_pM$, which depends smoothly on p.

c) Show that we have the following identities:

$$(\nabla_X R^{(0,4)})(Y, Z, U, W) = -(\nabla_X R^{(0,4)})(Z, Y, U, W)$$
$$= (\nabla_X R^{(0,4)})(U, W, Y, Z) = -(\nabla_X R^{(0,4)})(Y, Z, W, U)$$

d) Let $T \in \Gamma(T^{(0,s)}M)$ be a $(0,s)$-tensor for $s \geq 1$. We define the *divergence* of T by

$$\mathsf{div}(T)(X_1, \ldots, X_{s-1}) := \sum_{j=1}^{n} (\nabla_{e_j} T)(e_j, X_1, \ldots, X_{s-1}),$$

where $\{e_j\}_j$ is an orthonormal basis of T_pM and $X_1, \ldots, X_{s-1} \in T_pM$. Show:

$$\mathsf{div}(\mathrm{ric}) = \frac{1}{2} d\,\mathrm{scal}\,.$$

Hint: Use the second Bianchi identity for the Riemannian curvature tensor.

4. Exercise: *Schur's Lemma* (4 points).
Let (M^n, g) be a Riemannian manifold.

a) Assume $n \geq 2$ and the sectional curvature K_p only depends on the point p. Then Riemannian curvature tensor is of the form

$$g(R(X, Y)Z, W) = \kappa \cdot (g(X, Z)g(Y, W) - g(Y, Z)g(X, W))$$

where $\kappa\colon M \to \mathbb{R}$ is a smooth function.

b) Assume $n \geq 3$ and the Riemannian curvature tensor is of the form above. Show that $\mathrm{ric} = (n-1)\kappa g$ holds and that in this case that the function κ is locally constant. *Hint: Use Exercise* 3, *d*).