Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl
Please hand in the exercises until **Tuesday, December 19**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 9

1. Exercise (4 points).
Let (M^n, g) be a Riemannian manifold and $x: U \to V$ be a chart of M. Define

$$R^l_{ijk} = dx^l \left(R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \frac{\partial}{\partial x^k} \right)$$

the components of the Riemannian curvature tensor with respect to the chart x. Show that in these coordinate the representation of the curvature tensor in terms of the Christoffel symbols is given by:

$$R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_{m=1}^{n} \left(\Gamma^l_{mi} \Gamma^m_{kj} - \Gamma^l_{mj} \Gamma^m_{ki} \right).$$

2. Exercise (4 points).
Consider the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with induced Riemannian metric $g_{\mathbb{S}^n}$. Let $\{e_i\}_i \subset \mathbb{R}^{n+1}$ be the standard orthonormal basis and define the vector fields $X_i \in \mathfrak{X}(\mathbb{R}^{n+1})$

$$(X_i)\big|_p = e_i - \langle e_i, p \rangle p \text{ for all } p \in \mathbb{R}^{n+1}$$

In this exercise we want to compute the Riemannian curvature tensor of the standard metric of the sphere. We proceed as follows:

a) Show that $X_i\big|_{\mathbb{S}^n} \in \mathfrak{X}(\mathbb{S}^n)$.

b) Recall that the Levi-Civita connection on $\mathbb{S}^n$ is given by $(\nabla_X Y)_{|p} = \pi_p^{\tan}(\partial_X \tilde{Y})$ for $X \in T_pM$ and $Y \in \mathfrak{X}(\mathbb{S}^n)$ with an extension $\tilde{Y} \in \mathfrak{X}(\mathbb{R}^{n+1})$ and $\pi_p^{\tan}$ is the orthogonal projection $\mathbb{R}^{n+1} \to T_p\mathbb{S}^n$. Show:

$$(\nabla_{X_j} X_k)\big|_p = -\langle e_k, p \rangle X_j\big|_p$$

c) Show for $i, j, k \geq 2$: $(R(X_i, X_j)X_k)\big|_{e_1} = -\delta_{ik} e_j + \delta_{jk} e_i$.

d) Show that for all points $p, q \in \mathbb{S}^n$ there exists a $A \in \mathrm{SO}(n+1)$ such that $Ap = q$ holds. Conclude that the full Riemannian curvature of the standard sphere is given by:

$$g_{\mathbb{S}^n}(R(X,Y)Z,T) = g_{\mathbb{S}^n}(Y,Z) g_{\mathbb{S}^n}(X,T) - g_{\mathbb{S}^n}(X,Z) g_{\mathbb{S}^n}(Y,T).$$

3. Exercise (4 points).
Let (M, g) be a Riemannian manifold and $p \in M$ a point in M. Let $\hat{R}$ be a curvature tensor for T_pM, i.e. a tensor $\hat{R} \in T_pM \otimes (T_p^*M)^{\otimes 3}$, which satisfies the following identities:

$$\hat{R}(X_1, X_2, X_3) = -\hat{R}(X_2, X_1, X_3)$$
$$g_p(\hat{R}(X_1, X_2, X_3), X_4) = -g_p(\hat{R}(X_1, X_2, X_4), X_3)$$
$$\hat{R}(X_1, X_2, X_3) + \hat{R}(X_2, X_3, X_1) + \hat{R}(X_3, X_1, X_2) = 0$$

for all $X_1, X_2, X_3, X_4 \in T_pM$. We take a chart $x: U \to V$ with $x(p) = 0$ and construct a Riemannian metric

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{\alpha,\beta} \hat{R}_{i\alpha\beta j} x^\alpha x^\beta$$

on the chart neighborhood U. Show that $R_p = \hat{R}$ holds.

4. Exercise (4 points).
Let (M, g) be a Riemannian manifold and $f: M \to \mathbb{R}$ be a smooth function. We define *gradient vector field* of f by

$$g(\operatorname{grad} f, X) = X(f)$$

for all $X \in \mathfrak{X}(M)$. Moreover we define the *Hessian* of f by

$$\operatorname{Hess}(f)(X, Y) = (\nabla df)(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$.

a) Show that the gradient is a well-defined smooth vector field on M.

b) Let $x: U \to V$ be a chart. Show the local representation of the gradient of f:

$$\operatorname{grad} f|_U = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

If (e_i) is a generalized orthonormal basis of T_pM with $g_p(e_i, e_j) = \epsilon_i \delta_{ij}$, then show

$$\operatorname{grad} f|_p = \sum_i \epsilon_i \partial_{e_i} f \cdot e_i$$

c) Show that the Hessian of f is a well-defined $(0, 2)$ tensor on M. Does it depend on g?

d) Show that the Hessian is given by $\operatorname{Hess}(f) = \partial_X\big(\partial_Y(f)\big) - (\nabla_X Y)(f)$ and that $\operatorname{Hess}(f)$ is symmetric.