Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, December 12**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 8

1. Exercise (4 points).
Let $(M_1, g_1), (M_2, g_2)$ be two Riemannian manifolds with the induced Levi-Civita connections ∇^1, ∇^2. We identify (as in Exercise sheet no. 3, Exercise 1)

$$T_{(p,q)}(M_1 \times M_2) \cong T_p M_1 \times T_q M_2$$

and define the product metric $g_1 \oplus g_2$ on $M_1 \times M_2$ by

$$g_1 \oplus g_2 \left((v_1, w_1), (v_2, w_2)\right) = g_1(v_1, v_2) + g_2(w_1, w_2).$$

For vector fields $X_i \in \mathfrak{X}(M_i)$ where $i = 1, 2$ we define $X_1 \oplus X_2 \in \mathfrak{X}(M_1 \times M_2)$ by the formula

$$(X_1 \oplus X_2)|_{(p,q)} = \left(X_1|_p, 0|_q\right) + \left(0|_p, X_2|_q\right).$$

a) Construct a vector $X \in \mathfrak{X}(\mathbb{R}^2)$ that cannot be written as $X = X_1 \oplus X_2$ for vectors fields $X_i \in \mathfrak{X}(\mathbb{R})$.

b) Let $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ be vector fields on $M_1 \times M_2$. Show that the Levi-Civita connection ∇ of $(M_1 \times M_2, g_1 \oplus g_2)$ satisfies

$$\nabla_Y X = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2.$$

c) Let c_1, c_2 be geodesics on M_1 respectively M_2. Conclude, that $c(t) = (c_1(t), c_2(t))$ is a geodesic on $M_1 \times M_2$.

2. Exercise (4 points).
Consider the hyperbolic plane $(\mathfrak{H}, g^{\mathrm{hyp}})$, where

$$\mathfrak{H} = \left\{x + iy \in \mathbb{C} \mid x \in \mathbb{R} \text{ and } y > 0\right\}$$

with metric given by $g^{\mathrm{hyp}}_{x+iy} = \frac{1}{y^2} g^{\mathrm{eucl}}$. Let $r > 0, a \in \mathbb{R}$. Show that the half-circles

$$C_{r,a} = \left\{z \in \mathfrak{H} \mid |z - a| = r\right\}$$

are (up to reparametrisation) geodesics of the hyperbolic plane.
A way to solve this is as follows. First show that one can reduce to the case $(r, a) = (1, 0)$. *Then find a Möbius transformation* $\Psi_A \colon z \mapsto \frac{az+b}{cz+d}$ *where* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2; \mathbb{R})$, *with* $\Psi_A(i) = i$ *and* $\Psi_A(0) = -1$. *Conclude the statement by application of* Ψ_A *to the geodesic* $\gamma(t) = ie^t$.

3. Exercise: *Models of the hyperbolic plane* (4 points).
In this Exercise we want to identify three models of the hyperbolic plane.

- The *hyperboloid* model

$$\mathbb{H}^2 = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1 \text{ and } z > 0\right\}$$

  equipped with the induced metric from $\mathbb{R}^{2,1}$ (as in Sheet no. 7, Exercise 1).

- The *Poincaré half-plane* model

$$\mathfrak{H} = \left\{x + iy \in \mathbb{C} \mid y > 0\right\},$$

  with the Riemannian metric $g^{\mathfrak{H}}_{x+iy} = \frac{1}{y^2} g^{\text{eucl}}$.

- The *Poincaré disk* model

$$\mathbb{D} = \left\{x + iy \in \mathbb{C} \mid x^2 + y^2 < 1\right\},$$

  equipped with the metric $g^{\mathbb{D}}_{x+iy} = \left(\frac{2}{(1-(x^2+y^2))}\right)^2 g^{\text{eucl}}$.

a) We define a sterographic projection $f\colon \mathbb{H}^2 \to \mathbb{D}$ by the following procedure: Every point $p \in \mathbb{H}^2$ is send to the intersection point of the connecting straight line of p and the point $(0,0,-1)$ with the $x-y$-plane. Show that f is an isometry.

b) Show that the map

$$h : \mathfrak{H} \to \mathbb{D}, \quad z \mapsto \frac{z-i}{z+i}$$

is an isometry.

4. Exercise (4 points).
Let M and N be semi-Riemannian manifolds of the same dimension. Assume that N is connected.

a) Let $f_1, f_2\colon N \to M$ be two isometries. Assume there exists a point $p \in N$ such that $f_1(p) = f_2(p)$ and $d_p f_1 = d_p f_2$ holds. Show that the two isometries coincide.

b) Let $f\colon M \to M$ be an isometry. Show that the fix point set $\mathrm{Fix}(f) = \{p \in M \mid f(p) = p\}$ is a submanifold[1] of M.
Hint: Use the exponential function of M.

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[1]A subset $N \subset M^m$ of a smooth manifold M is a submanifold if for every point $p \in N$ there exists a chart $x\colon U \to V$ around the point p such that $x(U \cap N)$ is a submanifold of $\mathbb{R}^m$. Note that this definition does not exclude that different connected components might be of different dimension.