Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, December 5**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 7

1. Exercise (4 points).
We have already seen that

$$\mathbb{H}^n := \{X \in \mathbb{R}^{n,1} | \langle X, X \rangle = -1, X^{n+1} > 0\}$$

is a semi-Riemannian submanifold of $\mathbb{R}^{n,1}$. The induced Riemannian metric on $\mathbb{H}^n$ is called the hyperbolic metric g_{hyp}.

a) Let $f : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ be a linear map. Show that $f(e_1), \dots, f(e_{n+1})$ is a generalized o.n.b. iff f is an isometry. Show that $f(\mathbb{H}^n) = \mathbb{H}^n$ if f is an isometry with $\langle e_{n+1}, f(e_{n+1})\rangle_{n,1} < 0$.

b) Let $p, q \in \mathbb{H}^n, p \neq q$. Construct an isometry $f : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ such that $\operatorname{Fix}(f) = \operatorname{span}\{p, q\}$. Conclude that $f|_{\mathbb{H}^n}$ defines an isometry $\mathbb{H}^n \to \mathbb{H}^n$.

c) Define $\tilde{v} := q + \langle p, q\rangle_{n,1} \cdot p$ and $v := \tilde{v}/\sqrt{\langle \tilde{v}, \tilde{v}\rangle_{n,1}}$. Show that p, v is a generalized orthonormal basis of $\operatorname{span}\{p, q\}$. For $t \in \mathbb{R}$ we define $\gamma_{p,v}(t) := \cosh(t)p + \sinh(t)v$. Conclude that the image of $\gamma_{p,v}$ is $\mathbb{H}^n \cap \operatorname{span}\{p, q\}$.

d) Show that $\gamma_{p,v}$ is a geodesic. (Hint: Prop. 6.14 of the lecture can be helpful). Let γ be a geodesic in $\mathbb{H}^n$. Show that γ is either a constant or a reparametrisation of a $\gamma_{p,v}$ as above.

2. Exercise (4 points).
Let $F : M \to N$ be a smooth map between smooth manifolds M and N. Let X, Y (resp. $\tilde{X}, \tilde{Y}$) be (smooth) vector fields on M (resp. N). We say that X is F*-related* to $\tilde{X}$ if $dF \circ X = \tilde{X} \circ F$ holds on M.
Show that, if X is F-related to $\tilde{X}$ and Y is F-related to $\tilde{Y}$, then $[X, Y]$ is F-related to $[\tilde{X}, \tilde{Y}]$.

3. Exercise (4 points).
Let (M, g) be a semi-Riemannian manifold with Levi-Civita connection ∇.

i) Show that there exists a unique family of $\mathbb{R}$−bilinear operators

$$\nabla^{(r,s)} : \mathfrak{X}(M) \times \Gamma(T^{r,s}(M)) \to \Gamma(T^{r,s}(M)), \text{ where } r, s \in \mathbb{N}_0,$$

satisfying the following properties:

a) $\nabla_X^{(0,0)} f = \partial_X f$,
b) $\nabla_X^{(1,0)} Y = \nabla_X Y$,
c) $\left(\nabla_X^{(0,1)} \omega\right)(Y) = \partial_X(\omega(Y)) - \omega(\nabla_X Y)$,

d) $\nabla_X^{(r+r',s+s')}(T\otimes T') = \left(\nabla_X^{(r,s)}T\right)\otimes T' + T\otimes\left(\nabla_X^{(r',s')}T'\right).$

Hint: Show first that $\nabla^{(r,s)}$ *is a local operator and then construct it chartwise. Then check that on the intersection of the domains of two charts, the covariant derivations defined by the two charts coincide.*

Bonus: Show formally that this family of connections is C^∞–linear in the first argument:

$$\nabla_{fX}^{(r,s)}T = f\cdot\nabla_X^{(r,s)}T.$$

ii) Consider some tensor field $T\in\Gamma(T^{0,k}(M))$ with $k\in\mathbb{N}$. Show that for vector fields $X_1,\ldots,X_k\in\mathfrak{X}(M)$ one has the formula

$$\begin{aligned}\left(\nabla_X^{(0,k)}T\right)(X_1,\ldots,X_k) &= \partial_X\left(T(X_1,\ldots,X_k)\right)\\ &\quad-\sum_{i=1}^{k}T(X_1,\ldots,X_{i-1},\nabla_X X_i,X_{i+1},\ldots,X_k).\end{aligned}$$

4. Exercise (4 points).
Let (M,g) be a smooth compact Riemannian manifold. For $c\in\mathbb{R}_{>0}$ show that $S_cM := \{X\in TM \mid g(X,X)=c^2\}$ is compact. Then prove that every maximal geodesic of (M,g) is defined on all of $\mathbb{R}$.
Hint: recall what is known for maximally defined solutions of first order ODEs satisfying the Picard-Lindelöf assumptions on an open subsets of $\mathbb{R}^n$.