# Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, November 28**
**12 noon in the letterbox of your group (no. 15 or 16)**

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## Exercise Sheet no. 6

**1. Exercise** (4 points).
We define the hyperbolic plane as

$$\mathbb{H} := \{x + iy | x \in \mathbb{R}, y \in \mathbb{R}_{>0}\}$$

endowed with the metric $g^{\text{hyp}} := \frac{1}{y^2} g^{\text{eukl}}$ at $z = x + iy$.

a) Compute the Christoffel symbols with respect to the chart given by the identity $\mathbb{H} \to \mathbb{H} \subset \mathbb{R}^2$.

b) Compute explicitly the parallel transport $\mathcal{P}_{c_t,0,1} : T_{(0,1)}\mathbb{H} \to T_{(t,1)}\mathbb{H}$ along the curve $c_t : [0,1] \to \mathbb{H}$ with $c_t(s) := (st, 1)$.

c) Let $x_0 \in \mathbb{R}$ and $a \in \mathbb{R}\backslash\{0\}$. Show that $\gamma : \mathbb{R} \to \mathbb{H}, t \mapsto (x_0, e^{at})$ satisfies $\frac{\nabla}{dt}\dot{\gamma}(t) = 0$.

**2. Exercise** (4 points).
We consider $S^2 := \{p \in \mathbb{R}^3 \mid \|p\| = 1\}$ with the metric induced from $\mathbb{R}^3$.

a) We consider the following local parametrization

$$\psi : (0, 2\pi) \times (0, \pi) \to S^2, (\varphi, \theta) \mapsto (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))$$

whose inverse defines so-called *spherical polar coordinates*. We also write $x^1 = \phi$ and $x^2 = \theta$. Calculate the associated coordinate vector fields, the coefficients $g_{ij}$ of the metric and the Christoffel symbols $\Gamma_{ij}^k$.

b) For $\theta \in (0, \pi)$ we define $c: [0, 2\pi] \to S^2$, $c(t) := (\sin(\theta)\cos(t), \sin(\theta)\sin(t), \cos(\theta))$, $p := c(0)$. Compute the parallel transport $P_{c,0,2\pi}: T_pS^2 \to T_pS^2$, so $P_{c,0,2\pi} \in \text{End}(T_pS^2)$.

**3. Exercise**: *Levi–Civita connection for submanifolds* (4 points).
Assume $N, K \in \mathbb{N}_0$. Let $M$ be a semi-Riemannian submanifold of $\mathbb{R}^{N,K} = \left(\mathbb{R}^{N+K}, \langle\cdot,\cdot\rangle_{N,K}\right)$ where $\langle\cdot,\cdot\rangle_{N,K}$ was defined in Exercise 1 of Sheet no. 1. We write $\iota: M \to \mathbb{R}^{N+K}$ for the inclusion. Then for $p \in M$ we get an embedding $d_p\iota: T_pM \to \mathbb{R}^{N,K}$ which we use to identify $T_pM$ with its image in $\mathbb{R}^{N,K}$.

a) Show that there is a well-defined linear map

$$\pi_p^{\tan}: \mathbb{R}^{N,K} \to T_pM$$

that is the identity on $T_pM$ and such that

$$\ker(\pi_p^{\tan}) = \{X \in \mathbb{R}^{N,K} \mid \langle X, Y\rangle_{N,K} = 0 \,\forall Y \in T_pM\}.$$

Now let $X \in T_pM$ and let $Y \in \mathfrak{X}(M)$ be given. You may assume in this exercise that there is a smooth vector field $\widetilde{Y} \in \mathfrak{X}(\mathbb{R}^{N+K})$, $\widetilde{Y} = (\widetilde{Y}^1, \dots, \widetilde{Y}^{N+K})\colon \mathbb{R}^{N,K} \to \mathbb{R}^{N,K}$ such that

$$\widetilde{Y}|_M = Y\,.$$

Let $\partial_X \widetilde{Y}$ be defined componentwise, i.e. let $\partial_X \widetilde{Y} = (\partial_X \widetilde{Y}^1 \dots, \partial_X \widetilde{Y}^{N+K})$. We define $D_X \widetilde{Y} := \pi_p^{\tan}\big(\partial_X \widetilde{Y}\big)$. Prove the following:

b) $D_X \widetilde{Y}$ does not depend on how one extends $Y$ to $\widetilde{Y}$. Furthermore prove that $D_X \widetilde{Y}$ is local in the sense, that for a neighborhood $U \Subset \mathbb{R}^{N,K}$ of $p$, the term $D_X \widetilde{Y}$ only depends on $X$ and $Y|_{U \cap M}$.

c) Show that $D_X \widetilde{Y}$ satisfies the properties

- (ii) linearity in $\widetilde{Y}$
- (iv) product rule
- (v) metric compatibility

in the definition of the Levi–Civita connection in the lecture from Nov 10th given by M. Ludewig.

d) Let $\widetilde{X}\colon \mathbb{R}^{N,K} \to \mathbb{R}^{N,K}$ be a smooth extension of $X$ with $\forall_{q \in M} : \widetilde{X}|_q \in T_qM$. Show

$$D_X \widetilde{Y} - D_{Y|_p} \widetilde{X} = \big[\widetilde{X}, \widetilde{Y}\big]|_p\,.$$

e) Conclude that $D_X \widetilde{Y} = (\nabla_X Y)|_p$.

*As defined on Nov 10th, $\nabla$ denotes the Levi–Civita connection of the semi-Riemannian manifold $M$ in this formula.*

**4. Exercise** (4 points).
Let $M$ be a smooth, not necessarily compact, manifold. Given a 1-parameter group of diffeomorphisms $\varphi : M \times \mathbb{R} \to M$, $(x,t) \mapsto \varphi_t(x)$ on $M$, let $X$ be the associated tangent vector field on $M$ as in Exercise no. 3 of sheet 5. Show that, for any smooth tangent vector field $Y$ on $M$ and point $p \in M$ it is

$$\frac{d}{dt}\Big|_{t=0} ((\varphi_t)_* Y)\,\big|_p = -[X,Y]\big|_p,$$

where, for any diffeomorphism $\psi : M \to M$, the term $\psi_* Y$ denotes the pushforward tangent vector field of $Y$ defined by $\psi_* Y := \mathrm{d}\psi \circ Y \circ \psi^{-1}$.