Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, November 21**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 5

1. Exercise (4 points).
Let M be a smooth manifold and T a $C^\infty(M)$–linear map

$$T : \mathfrak{X}(M) \to C^\infty(M)$$

Show that there exists a unique smooth 1-form $\alpha \in C^\infty(M; T^*M)$ such that for all $X \in \mathfrak{X}(M)$ and for all $p \in M$ the equality

$$\big(T(X)\big)(p) = \alpha|_p(X|_p)$$

holds.
Hint: You may use without a proof that on a smooth manifold there is always a family of smooth functions $(\xi_i)_{i\in I}$ *such that* $(\eta_i := \xi_i^2)_{i\in I}$ *is a partition of unity.*

2. Exercise (4 points).
Let M be a smooth n–dimensional manifold and let Der^M be the space of derivations on M, that is, of all linear maps $\delta : C^\infty(M) \to C^\infty(M)$ which satisfy the following product rule:

$$\forall f_1, f_2 \in C^\infty(M) : \ \delta(f_1 f_2) = (\delta f_1) f_2 + f_1(\delta f_2).$$

It follows from the lecture (the results about derivations in a point $p \in M$) that the map

$$\mathfrak{X}(M) \to \mathrm{Der}^M, \ X \mapsto \partial_X$$

is well-defined and it can be checked that it is even an isomorphism.
Let X, Y now be two smooth tangent vector fields on M.

a) Show that $[\partial_X, \partial_Y] := \partial_X \circ \partial_Y - \partial_Y \circ \partial_X$ defines a derivation on M and deduce that there exists a unique smooth tangent vector field on M, which we denote by $[X, Y]$, such that $\partial_{[X,Y]} = [\partial_X, \partial_Y]$.

b) Show that, for any $f \in C^\infty(M)$, one has $[X, fY] = \partial_X f \cdot Y + f[X, Y]$.

c) Show that, if $x{:}U \to V$ is a chart of M, then $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ for all $1 \le i, j \le n$. Deduce that, if $X|_U = X^i \frac{\partial}{\partial x^i}$ and $Y|_U = Y^i \frac{\partial}{\partial x^i}$, then

$$[X, Y]|_U = \big(\partial_X(Y^i) - \partial_Y(X^i)\big)\frac{\partial}{\partial x^i} = \Big(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\Big)\frac{\partial}{\partial x^i}.$$

3. Exercise (4 points).
Let M be a compact smooth n-dimensional manifold. By definition, a *one-parameter group of diffeomorphisms* on M is a smooth map $\varphi : M \times \mathbb{R} \to M$, $(x,t) \mapsto \varphi_t(x)$, with $\varphi_0 = \mathrm{Id}_M$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $s,t \in \mathbb{R}$.

a) Show that, given any one-parameter group of diffeomorphisms $(\varphi_t)_t$ on M, the map $X|_x := \frac{d}{dt}|_{t=0}(\varphi_t(x))$ defines a smooth tangent vector field on M.

b) Prove that a one-parameter group of diffeomorphisms φ_t as above with $X \in \mathfrak{X}(M)$ as in a) necessarily has to satisfy

$$\frac{d}{dt}\Big|_{t=s}(\varphi_t(x)) = \mathrm{d}\varphi_s\big(X|_x\big) = X|_{\varphi_s(x)}\,.$$

c) Conversely, show that, given any smooth vector field X on M, there exists a unique one-parameter group of diffeomorphisms $(\varphi_t)_t$ on M such that $\frac{d}{dt}|_{t=0}(\varphi_t(x)) = X(x)$ for all $x \in M$.
Hint: First construct $\varphi_t(x)$ for fixed x and t close to 0 using the theorem of Picard-Lindelöf and using b); then show that $(x,t) \mapsto \varphi_t(x)$ can be extended to $M \times \mathbb{R}$.

4. Exercise: *Proof of Prop. II.4.7* (4 points).
Let N and M be smooth manifolds, and $\varphi\colon N \to M$ a smooth map, $p \in N$ and $\xi \in T_pN$. We equip M with a semi-Riemannian metric g, which then determines the Levi–Civita connection on M. Let $\eta, \tilde{\eta} \in C^\infty(N, \varphi^*TM)$ be two vector fields along φ. Show that

$$\partial_\xi\big(g(\eta,\tilde{\eta})\big) = g\big(\nabla_\xi \eta, \tilde{\eta}(p)\big) + g\big(\eta(p), \nabla_\xi \tilde{\eta}\big)\,.$$