Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, November 14**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 4

1. Exercise (4 points).

i) Let g be a symmetric bilinear form on a finite-dimensional vector space V, and let n_+, n_0 and n_- be the numbers of basis vectors $e_1, \dots, e_{n_+ + n_0 + n_-}$ with $g(e_i, e_i) = +1, 0$ or -1 as in Sylvester's law of inertia. Calculate

$$\max\left\{\dim W \;\middle|\; W \text{ is a linear subspace of } V \text{ on which } g \text{ is positive definite}\right\}$$
$$\max\left\{\dim W \;\middle|\; W \text{ is a linear subspace of } V \text{ on which } g \text{ is negative definite}\right\}$$
$$\max\left\{\dim W \;\middle|\; W \text{ is a linear subspace of } V \text{ on which } g \text{ is positive semi-definite}\right\}$$
$$\max\left\{\dim W \;\middle|\; W \text{ is a linear subspace of } V \text{ on which } g \text{ is negative semi-definite}\right\}$$
$$\max\left\{\dim W \;\middle|\; W \text{ is a linear subspace of } V \text{ with } g|_{W\times W} = 0\right\}$$

in terms of n_+, n_0 and n_-. Conclude that n_+, n_0 and n_- do not depend on the chosen basis.

ii) Let $B \in \mathbb{R}^{n\times n}$ be symmetric and $A \in \mathrm{GL}(n, \mathbb{R})$. Show that the numbers of positive, zero and negative eigenvalues of $A^\top B A$ does not depend on A.

2. Exercise (4 points).
Let $\mathcal{A} := \{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in A}$ be an atlas of an m−dimensional manifold M. Define for all $\alpha \in A$ the sets $U_\alpha^{TM} := \bigsqcup_{p\in U_\alpha} T_pM$ and the family $\mathcal{A}^{TM} = \{\mathrm{d}\varphi_\alpha : U_\alpha^{TM} \to V_\alpha \times \mathbb{R}^m\}_{\alpha\in A}$, where for a $v \in T_pM$ we set $\mathrm{d}\varphi_\alpha(v) := (p, \mathrm{d}_p\varphi_\alpha(v))$.

i) Show that TM carries a unique topology such that for all $\alpha \in A$ the subset U_α^{TM} is open and $\mathrm{d}\varphi_\alpha$ a homeomorphism.

ii) Show that TM with this topology is a topological manifold and $\mathcal{A}^{TM}$ a smooth atlas on TM.

iii) Show that $\pi : TM \to M$, $T_pM \ni v \mapsto p$ is a smooth map of manifolds.

iv) Show that some $X : M \to TM$ is smooth in the sense of the definition given in the lecture if and only if it is smooth as a map of manifolds $M \to TM$ and $\pi \circ X = \mathrm{id}_M$.

3. Exercise (4 points).
Let $W := \{p \in \mathbb{R}^3 | \max\{|p_1|, |p_2|, |p_3|\} = 1\}$.

i) Is W a submanifold of $\mathbb{R}^3$? Prove your statement.

ii) Equip W with the topology induced from $\mathbb{R}^3$ and show the existence of a C^∞−structure on W.

4. Exercise (4 points).
Let V be an n–dimensional vector space over $\mathbb{R}$.

i) Calculate $\dim(\Lambda^2 V) \otimes (\Lambda^2 V)$ and $\dim(\Lambda^3 V) \otimes V$.

ii) Show that

$$\begin{aligned} H : (\Lambda^2 V) \otimes (\Lambda^2 V) &\to (\Lambda^3 V) \otimes V \\ (x \wedge y) \otimes (z \wedge w) &\mapsto (x \wedge y \wedge z) \otimes w - (x \wedge y \wedge w) \otimes z \end{aligned}$$

is well-defined.

iii) Show that H is surjective and that $\dim \ker(H) = \frac{n^2(n^2-1)}{12}$.
Hint: Calculate $H((x \wedge y) \otimes (z \wedge w))$, $H((x \wedge z) \otimes (w \wedge y))$, and $H((x \wedge w) \otimes (y \wedge z))$ in order to show that $(x \wedge y \wedge z) \otimes w$ is in the image.