# Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, November 7**
**12 noon in the letterbox of your group (no. 15 or 16)**

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## Exercise Sheet no. 3

**1. Exercise** (4 points).
Let $M$ and $N$ be $m-$dimensional, resp. $n-$dimensional, $C^\infty-$manifolds with atlases

$$\mathcal{A}^M := \{x_i : U_i \to V_i\}_{i\in I} \text{ and } \mathcal{A}^N := \{y_j : U_j' \to V_j'\}_{j\in J}.$$

Define the family

$$\mathcal{A}^{M\times N} := \{z_{i,j} : U_i \times U_j' \to V_i \times V_j'\}_{(i,j)\in I\times J} \text{ with } z_{i,j}(p,q) := (x(p), y(q)).$$

a) Show that $\mathcal{A}^{M\times N}$ is a $C^\infty-$atlas on $M\times N$ with the product topology.

b) Equip $M\times N$ with the smooth structure defined by $\mathcal{A}^{M\times N}$ and show:

   i) The projection $\pi^M : M\times N \to M$ is $C^\infty$. (And, of course, so is $\pi^N$.)

   ii) For any smooth manifold $W$ and smooth maps $f : W \to M$ and $g : W \to N$ the map
   $$(f,g) : W \to M\times N \; p \mapsto (f(p), g(p))$$
   is smooth again.

c) Show that

$$T_{(p,q)}(M\times N) \to T_pM \times T_qN,\; X \mapsto (d_p\pi^M(X), d_q\pi^N(X))$$

is an isomorphism of vector spaces.

**2. Exercise** (4 points).
Let $k\in\mathbb{N}$ and $\epsilon > 0$ be given.

a) Define a diffeomorphism $F : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ such that $F$ restricted to $\mathbb{R}^{k+1}\setminus B_\epsilon(0)$ is the inclusion
$$\mathbb{R}^{k+1}\setminus B_\epsilon(0) \hookrightarrow \mathbb{R}^{k+1},$$
but $F(\mathbb{R}^k\times\{0\}) \not\subset \mathbb{R}^k\times\{0\}$.
*Hint: Use the graph of a function* $\eta:\mathbb{R}^k \to [0,\epsilon/4]$ *with support in* $\mathbb{R}^k\setminus B_{\epsilon/2}(0)$ *and use a function* $\chi:\mathbb{R}\to[0,1]$ *with support in* $(-\epsilon/2,\epsilon/2)$ *and some further properties.*

b) Show for all $m,n \geq 1$ that the atlas $\mathcal{A}^{M\times N}$ constructed in Exercise 1 is not a $C^\infty-$structure.

**3. Exercise** (4 points).
Viewing $\mathbb{Z}^n$ as a subgroup of $(\mathbb{R}^n, +)$ one obtains the quotient $T^n := \mathbb{R}^n/\mathbb{Z}^n$ (the $n$-dimensional torus) which, equipped with the quotient topology, is a topological manifold (you need not to prove this fact). Let $\pi : \mathbb{R}^n \to T^n$ be the projection.

a) Construct a $C^\infty$-atlas $= \{x_i : U_i \to V_i\}_{i \in I}$ on $T^n$ such that every $p \in \mathbb{R}^n$ has a neighbourhood $U$ that turns the restriction $\pi|_U : U \to \pi(U)$ into a diffeomorphism.

b) Show that $T^n$ is diffeomorphic to $\underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$.

c) Consider the submanifold

$$\mathbb{T} := \{(x, y, z)^T \in \mathbb{R}^3 | (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\}$$

of $\mathbb{R}^3$ which is obtained by rotating a circle in the halfplane $\{x > 0, y = 0\} \subset \mathbb{R}^3$ around the $z$-axis (you do not have to prove this).
Show that $T^2$ is diffeomorphic to $\mathbb{T}$.

**4. Exercise** (4 points).
Let $G$ be a $C^\infty$–manifold together with a smooth map $m : G \times G \to G$ such that $(G, m)$ is a group. In particular there is a neutral element $e \in G$.

a) Calculate
$$d_{(e,e)}m : T_{(e,e)}(G \times G)(\cong T_eG \times T_eG) \to T_eG.$$
*Hint: Calculate $d_{(e,e)}m(X, 0)$ and $d_{(e,e)}m(0, X)$ for $X \in T_eG$.*

b) Let $x : U \to V$ be a chart of $G$ with $e \in U$ and $x(e) = 0$. Let $U' \subset U$ be an open neighbourhood of $e$ such that $m(U' \times U') \subset U$. Denote $V' := x(U')$ and show that the differential of

$$F : V' \times V' \to V,\ (p, q) \mapsto x(m(x^{-1}(p), x^{-1}(q)))$$

is surjective in a neighbourhood of $0 \in V' \times V'$. *Hint: apply the implicit function theorem.*

c) Show that there is an open neighbourhood $W$ of $e$ and a smooth map $\mathrm{inv} : W \to G$ satisfying $m(p, \mathrm{inv}(p)) = e$ for $p \in W$. *Hint: Implicite function theorem.*

*Bonus:* Show that the map inv with its property in c) can be used to prove that $G \to G$, $g \mapsto g^{-1}$ is smooth. *Hint: Use $m(., g) : G \to G,\ g \in G$ to show smoothness on $m(W, g^{-1})$.*