Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, October 31**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 2

1. Exercise (4 points).
Let $k \in \mathbb{N} \cup \{0, \infty, \omega\}$.

a) Show that any C^k-atlas $\mathcal{A}$ is contained in exactly one C^k-structure $\overline{\mathcal{A}}$.
Hint: Define $\overline{\mathcal{A}}$ as the set of all charts that are C^k-compatible with all charts of $\mathcal{A}$. Then show the required properties.

b) Assume now $\mathcal{A}_1$ and $\mathcal{A}_2$ to be two C^k-atlases of M. Show that: $\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}$ if and only if all charts of $\mathcal{A}_1$ are C^k-compatible with all charts of $\mathcal{A}_2$.

2. Exercise (4 points).
We consider $\mathbb{R}$ with the standard topology, which is obviously a topological manifold. We consider four atlases $\mathcal{A}_{\text{std}}$, $\mathcal{A}_{\text{quad}}$, $\mathcal{A}_{\text{cub}}$, and $\mathcal{A}_{\text{unif}}$ on $\mathbb{R}$:

$$\mathcal{A}_{\text{std}} := \{(\text{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R})\}, \qquad \mathcal{A}_{\text{quad}} := \{(\text{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}), (\mathbb{R}_{>0} \to \mathbb{R}_{>0}, x \mapsto x^2)\}$$
$$\mathcal{A}_{\text{cub}} := \{(\mathbb{R} \to \mathbb{R}, x \mapsto x^3)\} \qquad \mathcal{A}_{\text{unif}} := \mathcal{A}_{\text{std}} \cup \mathcal{A}_{\text{cub}}$$

a) Determine for each atlas the maximal k such that it is a C^k-atlas.

b) Show that the C^1-structure defined by $\mathcal{A}_{\text{std}}$ is different from the C^1-structure defined by $\mathcal{A}_{\text{cub}}$. Are there two atlases among the four ones defined above, that define the same C^1-structure?

c) Construct a diffeomorphism $(\mathbb{R}, \mathcal{A}_{\text{std}}) \to (\mathbb{R}, \mathcal{A}_{\text{cub}})$.

3. Exercise (4 points).
We define a symmetric bilinear form $g^{(1,1)} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$g^{(1,1)}\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = xx' - yy' \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathbb{R}^2.$$

- Show that (b_1, b_2) is a generalized orthonormal basis for $g^{(1,1)}$ if and only if there exists a $t \in \mathbb{R}$ and $\delta, \epsilon \in \{1, -1\}$ such that

$$b_1 = \delta \cdot \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} \quad \text{and} \quad b_2 = \epsilon \cdot \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}.$$

- Determine the number of connected components of $O(1,1) := \text{Isom}_{\text{lin}}\left(\mathbb{R}^2, g^{(1,1)}\right)$.

4. Exercise (4 points).
Let $\mathbb{R}^{n\times n}_{\text{sym}} \subset \mathbb{R}^{n\times n}$ denote the subspace of symmetric $n \times n$-matrices.

a) Let $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{\text{sym}}$, $A \mapsto A^T A$, with A^T denoting matrix transposition. Show that $\mathbf{1}_n$ is a regular value for f.
Recall: Some c is by definition a regular value, if the differential $d_x f$ has full rank for all $x \in f^{-1}(c)$.

b) Determine $\ker(d_{\mathbf{1}_n} f)$.

c) Deduce that the orthogonal group $\mathrm{O}(n)$ is an $\frac{n(n-1)}{2}$-dimensional submanifold of $\mathbb{R}^{n^2} \cong \mathbb{R}^{n\times n}$.

d) Construct a chart of $\mathrm{O}(n)$ whose chart neighborhood contains $\mathbf{1}_n$.
Hint: Consider the exponential map $\exp(A) := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.