Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Please hand in the exercises until **Tuesday, October 24**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 1

1. Exercise (4 points).

i) Let $M := S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ be the n-sphere endowed with the topology induced by $\mathbb{R}^{n+1}$. Construct for any point $p \in S^n$ an open neighbourhood V of p in S^n and a homeomorphism from V to $\mathbb{R}^n$.

ii) On $\mathbb{R}^{n+k}$ define

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_{n+k} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{n+k} \end{pmatrix} \right\rangle_{n,k} := \sum_{i=1}^{n} x_i y_i - \sum_{i=n+1}^{n+k} x_i y_i.$$

Show for all $r \in \mathbb{R}\backslash\{0\}$, that $M := \{x \mid \langle x, x\rangle_{n,k} = r\}$ is a submanifold of $\mathbb{R}^{n+k}$.

2. Exercise (4 points).
On the set M we define the metric:

$$d : M \times M \to \mathbb{R}_{\geq 0},\ (x, y) \mapsto \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases},$$

inducing the discrete topology. Show that M is a Hausdorff space and locally Euclidean of some dimension $n \in \mathbb{N}_0$. What number is n? Show that the topology of M has a countable base, if and only if M is countable.

3. Exercise (4 points).
Let $n \in \mathbb{N}$ and $\mathbb{R}\mathrm{P}^n$ be the set of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$.

i) Identify $\mathbb{R}\mathrm{P}^n$ with the quotient $(\mathbb{R}^{n+1}\backslash\{0\})\backslash\sim$, where $x \sim y \iff \exists\, \lambda \in \mathbb{R}^{\times}$ s.t. $x = \lambda y$ and endow it with the quotient topology. Show that $\mathbb{R}\mathrm{P}^n$ is a compact Hausdorff space satisfying the second axiom of countability.
Hint for the Hausdorff property: You may use without a proof the triangle inequality for small angles, $\alpha_{x,z} \leq \alpha_{x,y} + \alpha_{y,z}$ where $\cos \alpha_{a,b} = \frac{\langle a,b\rangle}{\|a\|\|b\|}$.

ii) Show that the maps

$$U_j := \{[x] \in \mathbb{R}\mathrm{P}^n \mid x_j \neq 0\} \overset{\varphi_j}{\to} \mathbb{R}^n,\ [x] \mapsto \frac{1}{x_j}(x_1, \ldots, \widehat{x_j}, \ldots, x_{n+1}),\ 1 \leq j \leq n+1,$$

are well-defined homeomorphisms (the "$\widehat{x_j}$" means omitting "$, x_j,$").

iii) Show that $\mathcal{A} = (\phi_j : U_j \to \mathbb{R}^n)_{j \in \{1,2,\ldots,n+1\}}$ is an atlas for $\mathbb{R}\mathrm{P}^n$.

iv) For $i, j \in \{1, \ldots, n+1\}$, $i \neq j$ show that $\phi_j(U_i \cap U_j)$ is an open subset of $\mathbb{R}^n$ and that

$$\phi_i \circ (\phi_j)^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is a C^∞-diffeomorphism.

4. Exercise (4 points).
A topological space X is called *path-connected*, if any two points of X can be connected by a continuous path $\gamma : [0,1] \to X$. A topological space is called *locally path-connected*, if any neighbourhood of any point $x \in X$ contains a path-connected neighbourhood of x.

i) Show that any topological manifold is locally path-connected.

ii) Show that the connected components of a locally path-connected topological space are open and closed.

iii) Deduce that the connected components of an n-dimensional topological manifold are again n-dimensional topological manifolds.