Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
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Exercises for the week October 16 to 20

Recap Exercise Sheet

1. Exercise.

1.) A topological space X is called locally Euclidean of dimension $n \in \mathbb{N}$, if every $x \in X$ has an open neighbourhood U, such that U is homeomorphic to $\mathbb{R}^n$.

2.) A topological space X satisfies the second axiom of countability, if it has a countable basis of the topology (see e.g. section 1.1 in the script on Analysis IV by Prof. Garcke).

3.) A topological space X is called separable, if it contains a countable dense subset.

Let X be a locally Euclidean topological space satisfying the second axiom of countability.

i) Show that X can be covered by countably many neighbourhoods as in point 1.) above.

ii) Show that X is separable.

2. Exercise.
Let X be $\mathbb{R} \cup \{p\}$, where p is some object not contained in $\mathbb{R}$ and define

$$\mathcal{O} := \big\{U \mid U \text{ open in } \mathbb{R}\big\} \cup \big\{(U \setminus \{0\}) \cup \{p\} \mid U \text{ open in } \mathbb{R},\ 0 \in U\big\} \cup \big\{U \cup \{p\} \mid U \text{ open in } \mathbb{R},\ 0 \in U\big\}.$$

Show that $\mathcal{O}$ is a topology on X and prove that it is locally Euclidean, but not Hausdorff.

3. Exercise.
Let X be a topological space, $x \in X$. The connected component of x is defined as the union of all connected subsets of X containing x. Show that:

i) The connected component of x is connected.

ii) The connected component of x is closed in X.

4. Exercise.
Let X be a Hausdorff space such that every point in X has a compact neighbourhood. Show the following property (called local compactness): For any $x \in X$ and any neighbourhood U of x there is a compact neighbourhood of x contained in U.