Differential Geometry I: Exercises
University of Regensburg, Winter Term 2023/24
Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl
Exercises for the week October 16 to 20

Recap Exercise Sheet

1. Exercise.

1.) A topological space X is called locally Euclidean of dimension $n \in \mathbb{N}$, if every $x \in X$ has an open neighbourhood U, such that U is homeomorphic to $\mathbb{R}^n$.

2.) A topological space X satisfies the second axiom of countability, if it has a countable basis of the topology (see e.g. section 1.1 in the script on Analysis IV by Prof. Garcke).

3.) A topological space X is called separable, if it contains a countable dense subset.

Let X be a locally Euclidean topological space satisfying the second axiom of countability.

i) Show that X can be covered by countably many neighbourhoods as in point 1.) above.

ii) Show that X is separable.

2. Exercise.
Let X be $\mathbb{R} \cup \{p\}$, where p is some object not contained in $\mathbb{R}$ and define

$$\mathcal{O} := \left\{U \mid U \text{ open in } \mathbb{R}\right\} \cup \left\{(U \setminus \{0\}) \cup \{p\} \mid U \text{ open in } \mathbb{R},\ 0 \in U\right\} \cup \left\{U \cup \{p\} \mid U \text{ open in } \mathbb{R},\ 0 \in U\right\}.$$

Show that $\mathcal{O}$ is a topology on X and prove that it is locally Euclidean, but not Hausdorff.

3. Exercise.
Let X be a topological space, $x \in X$. The connected component of x is defined as the union of all connected subsets of X containing x. Show that:

i) The connected component of x is connected.

ii) The connected component of x is closed in X.

4. Exercise.
Let X be a Hausdorff space such that every point in X has a compact neighbourhood. Show the following property (called local compactness): For any $x \in X$ and any neighbourhood U of x there is a compact neighbourhood of x contained in U.

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Please hand in the exercises until **Tuesday, October 24**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 1

1. Exercise (4 points).

i) Let $M := S^n := \{x \in \mathbb{R}^{n+1} \,|\, \|x\| = 1\}$ be the n-sphere endowed with the topology induced by $\mathbb{R}^{n+1}$. Construct for any point $p \in S^n$ an open neighbourhood V of p in S^n and a homeomorphism from V to $\mathbb{R}^n$.

ii) On $\mathbb{R}^{n+k}$ define

$$\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_{n+k} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_{n+k} \end{pmatrix} \right\rangle_{n,k} := \sum_{i=1}^{n} x_i y_i - \sum_{i=n+1}^{n+k} x_i y_i.$$

Show for all $r \in \mathbb{R}\setminus\{0\}$, that $M := \{x \mid \langle x, x\rangle_{n,k} = r\}$ is a submanifold of $\mathbb{R}^{n+k}$.

2. Exercise (4 points).
On the set M we define the metric:

$$d : M \times M \to \mathbb{R}_{\geq 0},\ (x, y) \mapsto \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases},$$

inducing the discrete topology. Show that M is a Hausdorff space and locally Euclidean of some dimension $n \in \mathbb{N}_0$. What number is n? Show that the topology of M has a countable base, if and only if M is countable.

3. Exercise (4 points).
Let $n \in \mathbb{N}$ and $\mathbb{RP}^n$ be the set of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$.

i) Identify $\mathbb{RP}^n$ with the quotient $(\mathbb{R}^{n+1}\setminus\{0\})\setminus\sim$, where $x \sim y \iff \exists\, \lambda \in \mathbb{R}^\times$ s.t. $x = \lambda y$ and endow it with the quotient topology. Show that $\mathbb{RP}^n$ is a compact Hausdorff space satisfying the second axiom of countability.
Hint for the Hausdorff property: You may use without a proof the triangle inequality for small angles, $\alpha_{x,z} \leq \alpha_{x,y} + \alpha_{y,z}$ *where* $\cos\alpha_{a,b} = \frac{\langle a,b\rangle}{\|a\|\|b\|}$.

ii) Show that the maps

$$U_j := \{[x] \in \mathbb{RP}^n \,|\, x_j \neq 0\} \xrightarrow{\varphi_j} \mathbb{R}^n,\ [x] \mapsto \frac{1}{x_j}(x_1, \ldots, \widehat{x_j}, \ldots, x_{n+1}),\ 1 \leq j \leq n+1,$$

are well-defined homeomorphisms (the "$\widehat{x_j}$" means omitting "$, x_j,$").

iii) Show that $\mathcal{A} = (\phi_j : U_j \to \mathbb{R}^n)_{j \in \{1,2,\ldots,n+1\}}$ is an atlas for $\mathbb{RP}^n$.

iv) For $i, j \in \{1, \ldots, n+1\}$, $i \neq j$ show that $\phi_j(U_i \cap U_j)$ is an open subset of $\mathbb{R}^n$ and that

$$\phi_i \circ (\phi_j)^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is a C^∞-diffeomorphism.

4. Exercise (4 points).
A topological space X is called *path-connected*, if any two points of X can be connected by a continuous path $\gamma : [0,1] \to X$. A topological space is called *locally path-connected*, if any neighbourhood of any point $x \in X$ contains a path-connected neighbourhood of x.

i) Show that any topological manifold is locally path-connected.

ii) Show that the connected components of a locally path-connected topological space are open and closed.

iii) Deduce that the connected components of an n-dimensional topological manifold are again n-dimensional topological manifolds.

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Please hand in the exercises until **Tuesday, October 31**
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Exercise Sheet no. 2

1. Exercise (4 points).
Let $k \in \mathbb{N} \cup \{0, \infty, \omega\}$.

a) Show that any C^k-atlas $\mathcal{A}$ is contained in exactly one C^k-structure $\overline{\mathcal{A}}$.
Hint: Define $\overline{\mathcal{A}}$ as the set of all charts that are C^k-compatible with all charts of $\mathcal{A}$. Then show the required properties.

b) Assume now $\mathcal{A}_1$ and $\mathcal{A}_2$ to be two C^k-atlases of M. Show that: $\overline{\mathcal{A}_1} = \overline{\mathcal{A}_2}$ if and only if all charts of $\mathcal{A}_1$ are C^k-compatible with all charts of $\mathcal{A}_2$.

2. Exercise (4 points).
We consider $\mathbb{R}$ with the standard topology, which is obviously a topological manifold. We consider four atlases $\mathcal{A}_{\text{std}}$, $\mathcal{A}_{\text{quad}}$, $\mathcal{A}_{\text{cub}}$, and $\mathcal{A}_{\text{unif}}$ on $\mathbb{R}$:

$$\mathcal{A}_{\text{std}} := \{(\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R})\}, \qquad \mathcal{A}_{\text{quad}} := \{(\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}), (\mathbb{R}_{>0} \to \mathbb{R}_{>0}, x \mapsto x^2)\}$$
$$\mathcal{A}_{\text{cub}} := \{(\mathbb{R} \to \mathbb{R}, x \mapsto x^3)\} \qquad \mathcal{A}_{\text{unif}} := \mathcal{A}_{\text{std}} \cup \mathcal{A}_{\text{cub}}$$

a) Determine for each atlas the maximal k such that it is a C^k-atlas.

b) Show that the C^1-structure defined by $\mathcal{A}_{\text{std}}$ is different from the C^1-structure defined by $\mathcal{A}_{\text{cub}}$. Are there two atlases among the four ones defined above, that define the same C^1-structure?

c) Construct a diffeomorphism $(\mathbb{R}, \mathcal{A}_{\text{std}}) \to (\mathbb{R}, \mathcal{A}_{\text{cub}})$.

3. Exercise (4 points).
We define a symmetric bilinear form $g^{(1,1)} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by setting

$$g^{(1,1)}\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = xx' - yy' \quad \text{for all} \quad \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix} \in \mathbb{R}^2.$$

- Show that (b_1, b_2) is a generalized orthonormal basis for $g^{(1,1)}$ if and only if there exists a $t \in \mathbb{R}$ and $\delta, \epsilon \in \{1, -1\}$ such that

$$b_1 = \delta \cdot \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} \quad \text{and} \quad b_2 = \epsilon \cdot \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}.$$

- Determine the number of connected components of $O(1,1) := \mathrm{Isom}_{\mathrm{lin}}\left(\mathbb{R}^2, g^{(1,1)}\right)$.

4. Exercise (4 points).
Let $\mathbb{R}^{n\times n}_{\text{sym}} \subset \mathbb{R}^{n\times n}$ denote the subspace of symmetric $n \times n$-matrices.

a) Let $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{\text{sym}}$, $A \mapsto A^T A$, with A^T denoting matrix transposition. Show that $\mathbf{1}_n$ is a regular value for f.
Recall: Some c is by definition a regular value, if the differential $d_x f$ has full rank for all $x \in f^{-1}(c)$.

b) Determine $\ker(d_{\mathbf{1}_n} f)$.

c) Deduce that the orthogonal group $\mathrm{O}(n)$ is an $\frac{n(n-1)}{2}$-dimensional submanifold of $\mathbb{R}^{n^2} \cong \mathbb{R}^{n\times n}$.

d) Construct a chart of $\mathrm{O}(n)$ whose chart neighborhood contains $\mathbf{1}_n$.
Hint: Consider the exponential map $\exp(A) := \sum_{n=0}^{\infty} \frac{1}{n!} A^n$.

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Please hand in the exercises until **Tuesday, November 7**
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Exercise Sheet no. 3

1. Exercise (4 points).
Let M and N be m–dimensional, resp. n–dimensional, C^∞–manifolds with atlases

$$\mathcal{A}^M := \{x_i : U_i \to V_i\}_{i\in I} \text{ and } \mathcal{A}^N := \{y_j : U'_j \to V'_j\}_{j\in J}.$$

Define the family

$$\mathcal{A}^{M\times N} := \{z_{i,j} : U_i \times U'_j \to V_i \times V'_j\}_{(i,j)\in I\times J} \text{ with } z_{i,j}(p,q) := (x(p), y(q)).$$

a) Show that $\mathcal{A}^{M\times N}$ is a C^∞–atlas on $M \times N$ with the product topology.

b) Equip $M \times N$ with the smooth structure defined by $\mathcal{A}^{M\times N}$ and show:

   i) The projection $\pi^M : M \times N \to M$ is C^∞. (And, of course, so is π^N.)

   ii) For any smooth manifold W and smooth maps $f : W \to M$ and $g : W \to N$ the map
   $$(f, g) : W \to M \times N \; p \mapsto (f(p), g(p))$$
   is smooth again.

c) Show that
$$T_{(p,q)}(M \times N) \to T_pM \times T_qN, \; X \mapsto (d_p\pi^M(X), d_q\pi^N(X))$$
is an isomorphism of vector spaces.

2. Exercise (4 points).
Let $k \in \mathbb{N}$ and $\epsilon > 0$ be given.

a) Define a diffeomorphism $F : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ such that F restricted to $\mathbb{R}^{k+1} \smallsetminus B_\epsilon(0)$ is the inclusion
$$\mathbb{R}^{k+1} \smallsetminus B_\epsilon(0) \hookrightarrow \mathbb{R}^{k+1},$$
but $F(\mathbb{R}^k \times \{0\}) \not\subset \mathbb{R}^k \times \{0\}$.
Hint: Use the graph of a function $\eta: \mathbb{R}^k \to [0, \epsilon/4]$ *with support in* $\mathbb{R}^k \smallsetminus B_{\epsilon/2}(0)$ *and use a function* $\chi: \mathbb{R} \to [0, 1]$ *with support in* $(-\epsilon/2, \epsilon/2)$ *and some further properties.*

b) Show for all $m, n \geq 1$ that the atlas $\mathcal{A}^{M\times N}$ constructed in Exercise 1 is not a C^∞–structure.

3. Exercise (4 points).
Viewing $\mathbb{Z}^n$ as a subgroup of $(\mathbb{R}^n, +)$ one obtains the quotient $T^n := \mathbb{R}^n/\mathbb{Z}^n$ (the n-dimensional torus) which, equipped with the quotient topology, is a topological manifold (you need not to prove this fact). Let $\pi : \mathbb{R}^n \to T^n$ be the projection.

a) Construct a C^∞-atlas $= \{x_i : U_i \to V_i\}_{i \in I}$ on T^n such that every $p \in \mathbb{R}^n$ has a neighbourhood U that turns the restriction $\pi|_U : U \to \pi(U)$ into a diffeomorphism.

b) Show that T^n is diffeomorphic to $\underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$.

c) Consider the submanifold

$$\mathbb{T} := \{(x, y, z)^T \in \mathbb{R}^3 | (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\}$$

of $\mathbb{R}^3$ which is obtained by rotating a circle in the halfplane $\{x > 0, y = 0\} \subset \mathbb{R}^3$ around the z-axis (you do not have to prove this).
Show that T^2 is diffeomorphic to $\mathbb{T}$.

4. Exercise (4 points).
Let G be a C^∞–manifold together with a smooth map $m : G \times G \to G$ such that (G, m) is a group. In particular there is a neutral element $e \in G$.

a) Calculate
$$d_{(e,e)}m : T_{(e,e)}(G \times G)(\cong T_eG \times T_eG) \to T_eG.$$
Hint: Calculate $d_{(e,e)}m(X, 0)$ and $d_{(e,e)}m(0, X)$ for $X \in T_eG$.

b) Let $x : U \to V$ be a chart of G with $e \in U$ and $x(e) = 0$. Let $U' \subset U$ be an open neighbourhood of e such that $m(U' \times U') \subset U$. Denote $V' := x(U')$ and show that the differential of

$$F : V' \times V' \to V, \; (p, q) \mapsto x(m(x^{-1}(p), x^{-1}(q)))$$

is surjective in a neighbourhood of $0 \in V' \times V'$. *Hint: apply the implicit function theorem.*

c) Show that there is an open neighbourhood W of e and a smooth map $\mathrm{inv} : W \to G$ satisfying $m(p, \mathrm{inv}(p)) = e$ for $p \in W$. *Hint: Implicite function theorem.*

Bonus: Show that the map inv with its property in c) can be used to prove that $G \to G$, $g \mapsto g^{-1}$ is smooth. *Hint: Use $m(., g) : G \to G$, $g \in G$ to show smoothness on $m(W, g^{-1})$.*

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Exercise Sheet no. 4

1. Exercise (4 points).

i) Let g be a symmetric bilinear form on a finite-dimensional vector space V, and let n_+, n_0 and n_- be the numbers of basis vectors $e_1, \dots, e_{n_+ + n_0 + n_-}$ with $g(e_i, e_i) = +1, 0$ or -1 as in Sylvester's law of inertia. Calculate

$$\max\left\{\dim W \;\middle|\; W \text{ is a linear subspace of } V \text{ on which } g \text{ is positive definite}\right\}$$
$$\max\left\{\dim W \;\middle|\; W \text{ is a linear subspace of } V \text{ on which } g \text{ is negative definite}\right\}$$
$$\max\left\{\dim W \;\middle|\; W \text{ is a linear subspace of } V \text{ on which } g \text{ is positive semi-definite}\right\}$$
$$\max\left\{\dim W \;\middle|\; W \text{ is a linear subspace of } V \text{ on which } g \text{ is negative semi-definite}\right\}$$
$$\max\left\{\dim W \;\middle|\; W \text{ is a linear subspace of } V \text{ with } g|_{W\times W} = 0\right\}$$

in terms of n_+, n_0 and n_-. Conclude that n_+, n_0 and n_- do not depend on the chosen basis.

ii) Let $B \in \mathbb{R}^{n\times n}$ be symmetric and $A \in \mathrm{GL}(n, \mathbb{R})$. Show that the numbers of positive, zero and negative eigenvalues of $A^\top B A$ does not depend on A.

2. Exercise (4 points).
Let $\mathcal{A} := \{\varphi_\alpha : U_\alpha \to V_\alpha\}_{\alpha\in A}$ be an atlas of an m−dimensional manifold M. Define for all $\alpha \in A$ the sets $U_\alpha^{TM} := \bigsqcup_{p\in U_\alpha} T_pM$ and the family $\mathcal{A}^{TM} = \{\mathrm{d}\varphi_\alpha : U_\alpha^{TM} \to V_\alpha \times \mathbb{R}^m\}_{\alpha\in A}$, where for a $v \in T_pM$ we set $\mathrm{d}\varphi_\alpha(v) := (p, \mathrm{d}_p\varphi_\alpha(v))$.

i) Show that TM carries a unique topology such that for all $\alpha \in A$ the subset U_α^{TM} is open and $\mathrm{d}\varphi_\alpha$ a homeomorphism.

ii) Show that TM with this topology is a topological manifold and $\mathcal{A}^{TM}$ a smooth atlas on TM.

iii) Show that $\pi : TM \to M$, $T_pM \ni v \mapsto p$ is a smooth map of manifolds.

iv) Show that some $X : M \to TM$ is smooth in the sense of the definition given in the lecture if and only if it is smooth as a map of manifolds $M \to TM$ and $\pi \circ X = \mathrm{id}_M$.

3. Exercise (4 points).
Let $W := \{p \in \mathbb{R}^3 | \max\{|p_1|, |p_2|, |p_3|\} = 1\}$.

i) Is W a submanifold of $\mathbb{R}^3$? Prove your statement.

ii) Equip W with the topology induced from $\mathbb{R}^3$ and show the existence of a C^∞−structure on W.

4. Exercise (4 points).
Let V be an n−dimensional vector space over $\mathbb{R}$.

i) Calculate $\dim(\Lambda^2 V) \otimes (\Lambda^2 V)$ and $\dim(\Lambda^3 V) \otimes V$.

ii) Show that

$$\begin{aligned} H : (\Lambda^2 V) \otimes (\Lambda^2 V) &\to (\Lambda^3 V) \otimes V \\ (x \wedge y) \otimes (z \wedge w) &\mapsto (x \wedge y \wedge z) \otimes w - (x \wedge y \wedge w) \otimes z \end{aligned}$$

is well-defined.

iii) Show that H is surjective and that $\dim \ker(H) = \frac{n^2(n^2-1)}{12}$.
Hint: Calculate $H((x \wedge y) \otimes (z \wedge w))$, $H((x \wedge z) \otimes (w \wedge y))$, and $H((x \wedge w) \otimes (y \wedge z))$ in order to show that $(x \wedge y \wedge z) \otimes w$ is in the image.

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Exercise Sheet no. 5

1. Exercise (4 points).
Let M be a smooth manifold and T a $C^\infty(M)$–linear map

$$T : \mathfrak{X}(M) \to C^\infty(M)$$

Show that there exists a unique smooth 1-form $\alpha \in C^\infty(M; T^*M)$ such that for all $X \in \mathfrak{X}(M)$ and for all $p \in M$ the equality

$$\big(T(X)\big)(p) = \alpha|_p(X|_p)$$

holds.
Hint: You may use without a proof that on a smooth manifold there is always a family of smooth functions $(\xi_i)_{i\in I}$ such that $(\eta_i := \xi_i^2)_{i\in I}$ is a partition of unity.

2. Exercise (4 points).
Let M be a smooth n–dimensional manifold and let Der^M be the space of derivations on M, that is, of all linear maps $\delta : C^\infty(M) \to C^\infty(M)$ which satisfy the following product rule:

$$\forall f_1, f_2 \in C^\infty(M) : \ \delta(f_1 f_2) = (\delta f_1) f_2 + f_1(\delta f_2).$$

It follows from the lecture (the results about derivations in a point $p \in M$) that the map

$$\mathfrak{X}(M) \to \mathrm{Der}^M, \ X \mapsto \partial_X$$

is well-defined and it can be checked that it is even an isomorphism.
Let X, Y now be two smooth tangent vector fields on M.

a) Show that $[\partial_X, \partial_Y] := \partial_X \circ \partial_Y - \partial_Y \circ \partial_X$ defines a derivation on M and deduce that there exists a unique smooth tangent vector field on M, which we denote by $[X, Y]$, such that $\partial_{[X,Y]} = [\partial_X, \partial_Y]$.

b) Show that, for any $f \in C^\infty(M)$, one has $[X, fY] = \partial_X f \cdot Y + f[X, Y]$.

c) Show that, if $x\colon U \to V$ is a chart of M, then $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$ for all $1 \le i, j \le n$. Deduce that, if $X|_U = X^i \frac{\partial}{\partial x^i}$ and $Y|_U = Y^i \frac{\partial}{\partial x^i}$, then

$$[X, Y]|_U = \big(\partial_X(Y^i) - \partial_Y(X^i)\big)\frac{\partial}{\partial x^i} = \Big(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\Big)\frac{\partial}{\partial x^i}.$$

3. Exercise (4 points).
Let M be a compact smooth n-dimensional manifold. By definition, a *one-parameter group of diffeomorphisms* on M is a smooth map $\varphi : M \times \mathbb{R} \to M$, $(x,t) \mapsto \varphi_t(x)$, with $\varphi_0 = \mathrm{Id}_M$ and $\varphi_t \circ \varphi_s = \varphi_{t+s}$ for all $s,t \in \mathbb{R}$.

a) Show that, given any one-parameter group of diffeomorphisms $(\varphi_t)_t$ on M, the map $X|_x := \frac{d}{dt}|_{t=0}(\varphi_t(x))$ defines a smooth tangent vector field on M.

b) Prove that a one-parameter group of diffeomorphisms φ_t as above with $X \in \mathfrak{X}(M)$ as in a) necessarily has to satisfy

$$\frac{d}{dt}\Big|_{t=s}(\varphi_t(x)) = \mathrm{d}\varphi_s\big(X|_x\big) = X|_{\varphi_s(x)}\,.$$

c) Conversely, show that, given any smooth vector field X on M, there exists a unique one-parameter group of diffeomorphisms $(\varphi_t)_t$ on M such that $\frac{d}{dt}|_{t=0}(\varphi_t(x)) = X(x)$ for all $x \in M$.
Hint: First construct $\varphi_t(x)$ for fixed x and t close to 0 using the theorem of Picard-Lindelöf and using b); then show that $(x,t) \mapsto \varphi_t(x)$ can be extended to $M \times \mathbb{R}$.

4. Exercise: *Proof of Prop. II.4.7* (4 points).
Let N and M be smooth manifolds, and $\varphi\colon N \to M$ a smooth map, $p \in N$ and $\xi \in T_pN$. We equip M with a semi-Riemannian metric g, which then determines the Levi–Civita connection on M. Let $\eta, \tilde{\eta} \in C^\infty(N, \varphi^*TM)$ be two vector fields along φ. Show that

$$\partial_\xi\big(g(\eta,\tilde{\eta})\big) = g\big(\nabla_\xi \eta, \tilde{\eta}(p)\big) + g\big(\eta(p), \nabla_\xi \tilde{\eta}\big)\,.$$

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Exercise Sheet no. 6

1. Exercise (4 points).
We define the hyperbolic plane as

$$\mathbb{H} := \{x + iy | x \in \mathbb{R}, y \in \mathbb{R}_{>0}\}$$

endowed with the metric $g^{\text{hyp}} := \frac{1}{y^2} g^{\text{eukl}}$ at $z = x + iy$.

a) Compute the Christoffel symbols with respect to the chart given by the identity $\mathbb{H} \to \mathbb{H} \subset \mathbb{R}^2$.

b) Compute explicitly the parallel transport $\mathcal{P}_{c_t,0,1} : T_{(0,1)}\mathbb{H} \to T_{(t,1)}\mathbb{H}$ along the curve $c_t : [0,1] \to \mathbb{H}$ with $c_t(s) := (st, 1)$.

c) Let $x_0 \in \mathbb{R}$ and $a \in \mathbb{R}\backslash\{0\}$. Show that $\gamma : \mathbb{R} \to \mathbb{H}, t \mapsto (x_0, e^{at})$ satisfies $\frac{\nabla}{dt}\dot{\gamma}(t) = 0$.

2. Exercise (4 points).
We consider $S^2 := \{p \in \mathbb{R}^3 \mid \|p\| = 1\}$ with the metric induced from $\mathbb{R}^3$.

a) We consider the following local parametrization

$$\psi : (0, 2\pi) \times (0, \pi) \to S^2, (\varphi, \theta) \mapsto (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))$$

whose inverse defines so-called *spherical polar coordinates*. We also write $x^1 = \phi$ and $x^2 = \theta$. Calculate the associated coordinate vector fields, the coefficients g_{ij} of the metric and the Christoffel symbols Γ_{ij}^k.

b) For $\theta \in (0, \pi)$ we define $c: [0, 2\pi] \to S^2$, $c(t) := (\sin(\theta)\cos(t), \sin(\theta)\sin(t), \cos(\theta))$, $p := c(0)$. Compute the parallel transport $P_{c,0,2\pi}: T_pS^2 \to T_pS^2$, so $P_{c,0,2\pi} \in \text{End}(T_pS^2)$.

3. Exercise: *Levi–Civita connection for submanifolds* (4 points).
Assume $N, K \in \mathbb{N}_0$. Let M be a semi-Riemannian submanifold of $\mathbb{R}^{N,K} = \left(\mathbb{R}^{N+K}, \langle \cdot, \cdot \rangle_{N,K}\right)$ where $\langle \cdot, \cdot \rangle_{N,K}$ was defined in Exercise 1 of Sheet no. 1. We write $\iota: M \to \mathbb{R}^{N+K}$ for the inclusion. Then for $p \in M$ we get an embedding $d_p\iota: T_pM \to \mathbb{R}^{N,K}$ which we use to identify T_pM with its image in $\mathbb{R}^{N,K}$.

a) Show that there is a well-defined linear map

$$\pi_p^{\tan}: \mathbb{R}^{N,K} \to T_pM$$

that is the identity on T_pM and such that

$$\ker(\pi_p^{\tan}) = \{X \in \mathbb{R}^{N,K} \mid \langle X, Y \rangle_{N,K} = 0 \, \forall Y \in T_pM\}.$$

Now let $X \in T_pM$ and let $Y \in \mathfrak{X}(M)$ be given. You may assume in this exercise that there is a smooth vector field $\widetilde{Y} \in \mathfrak{X}(\mathbb{R}^{N+K})$, $\widetilde{Y} = (\widetilde{Y}^1, \dots, \widetilde{Y}^{N+K}) \colon \mathbb{R}^{N,K} \to \mathbb{R}^{N,K}$ such that

$$\widetilde{Y}|_M = Y\,.$$

Let $\partial_X \widetilde{Y}$ be defined componentwise, i.e. let $\partial_X \widetilde{Y} = (\partial_X \widetilde{Y}^1 \dots, \partial_X \widetilde{Y}^{N+K})$. We define $D_X \widetilde{Y} := \pi_p^{\tan}\big(\partial_X \widetilde{Y}\big)$. Prove the following:

b) $D_X\widetilde{Y}$ does not depend on how one extends Y to $\widetilde{Y}$. Furthermore prove that $D_X\widetilde{Y}$ is local in the sense, that for a neighborhood $U \Subset \mathbb{R}^{N,K}$ of p, the term $D_X\widetilde{Y}$ only depends on X and $Y|_{U\cap M}$.

c) Show that $D_X\widetilde{Y}$ satisfies the properties

- (ii) linearity in $\widetilde{Y}$
- (iv) product rule
- (v) metric compatibility

in the definition of the Levi–Civita connection in the lecture from Nov 10th given by M. Ludewig.

d) Let $\widetilde{X} \colon \mathbb{R}^{N,K} \to \mathbb{R}^{N,K}$ be a smooth extension of X with $\forall_{q\in M} : \widetilde{X}|_q \in T_qM$. Show

$$D_X\widetilde{Y} - D_{Y|_p}\widetilde{X} = \big[\widetilde{X}, \widetilde{Y}\big]|_p\,.$$

e) Conclude that $D_X\widetilde{Y} = (\nabla_X Y)|_p$.

As defined on Nov 10th, ∇ denotes the Levi–Civita connection of the semi-Riemannian manifold M in this formula.

4. Exercise (4 points).
Let M be a smooth, not necessarily compact, manifold. Given a 1-parameter group of diffeomorphisms $\varphi : M \times \mathbb{R} \to M$, $(x,t) \mapsto \varphi_t(x)$ on M, let X be the associated tangent vector field on M as in Exercise no. 3 of sheet 5. Show that, for any smooth tangent vector field Y on M and point $p \in M$ it is

$$\frac{d}{dt}\Big|_{t=0} ((\varphi_t)_* Y)\,\Big|_p = -[X,Y]\Big|_p,$$

where, for any diffeomorphism $\psi : M \to M$, the term $\psi_* Y$ denotes the pushforward tangent vector field of Y defined by $\psi_* Y := \mathrm{d}\psi \circ Y \circ \psi^{-1}$.

Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl
Please hand in the exercises until **Tuesday, December 5**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 7

1. Exercise (4 points).
We have already seen that

$$\mathbb{H}^n := \{X \in \mathbb{R}^{n,1} | \langle X, X\rangle = -1, X^{n+1} > 0\}$$

is a semi-Riemannian submanifold of $\mathbb{R}^{n,1}$. The induced Riemannian metric on $\mathbb{H}^n$ is called the hyperbolic metric g_{hyp}.

a) Let $f : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ be a linear map. Show that $f(e_1), \ldots, f(e_{n+1})$ is a generalized o.n.b. iff f is an isometry. Show that $f(\mathbb{H}^n) = \mathbb{H}^n$ if f is an isometry with $\langle e_{n+1}, f(e_{n+1})\rangle_{n,1} < 0$.

b) Let $p, q \in \mathbb{H}^n, p \neq q$. Construct an isometry $f : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ such that $\text{Fix}(f) = \text{span}\{p, q\}$. Conclude that $f|_{\mathbb{H}^n}$ defines an isometry $\mathbb{H}^n \to \mathbb{H}^n$.

c) Define $\tilde{v} := q + \langle p, q\rangle_{n,1} \cdot p$ and $v := \tilde{v}/\sqrt{\langle \tilde{v}, \tilde{v}\rangle_{n,1}}$. Show that p, v is a generalized orthonormal basis of $\text{span}\{p, q\}$. For $t \in \mathbb{R}$ we define $\gamma_{p,v}(t) := \cosh(t)p + \sinh(t)v$. Conclude that the image of $\gamma_{p,v}$ is $\mathbb{H}^n \cap \text{span}\{p, q\}$.

d) Show that $\gamma_{p,v}$ is a geodesic. (Hint: Prop. 6.14 of the lecture can be helpful). Let γ be a geodesic in $\mathbb{H}^n$. Show that γ is either a constant or a reparametrisation of a $\gamma_{p,v}$ as above.

2. Exercise (4 points).
Let $F : M \to N$ be a smooth map between smooth manifolds M and N. Let X, Y (resp. $\tilde{X}, \tilde{Y}$) be (smooth) vector fields on M (resp. N). We say that X is F-*related* to $\tilde{X}$ if $dF \circ X = \tilde{X} \circ F$ holds on M.
Show that, if X is F-related to $\tilde{X}$ and Y is F-related to $\tilde{Y}$, then $[X, Y]$ is F-related to $[\tilde{X}, \tilde{Y}]$.

3. Exercise (4 points).
Let (M, g) be a semi-Riemannian manifold with Levi-Civita connection ∇.

i) Show that there exists a unique family of $\mathbb{R}$−bilinear operators

$$\nabla^{(r,s)} : \mathfrak{X}(M) \times \Gamma(T^{r,s}(M)) \to \Gamma(T^{r,s}(M)), \text{ where } r, s \in \mathbb{N}_0,$$

satisfying the following properties:

a) $\nabla^{(0,0)}_X f = \partial_X f$,
b) $\nabla^{(1,0)}_X Y = \nabla_X Y$,
c) $\left(\nabla^{(0,1)}_X \omega\right)(Y) = \partial_X(\omega(Y)) - \omega(\nabla_X Y)$,

d) $\nabla_X^{(r+r',s+s')}(T \otimes T') = \left(\nabla_X^{(r,s)} T\right) \otimes T' + T \otimes \left(\nabla_X^{(r',s')} T'\right).$

Hint: Show first that $\nabla^{(r,s)}$ is a local operator and then construct it chartwise. Then check that on the intersection of the domains of two charts, the covariant derivations defined by the two charts coincide.

Bonus: Show formally that this family of connections is C^∞−linear in the first argument:

$$\nabla_{fX}^{(r,s)} T = f \cdot \nabla_X^{(r,s)} T.$$

ii) Consider some tensor field $T \in \Gamma(T^{0,k}(M))$ with $k \in \mathbb{N}$. Show that for vector fields $X_1, \ldots, X_k \in \mathfrak{X}(M)$ one has the formula

$$\begin{aligned}\left(\nabla_X^{(0,k)} T\right)(X_1, \ldots, X_k) &= \partial_X \left(T(X_1, \ldots, X_k)\right) \\ &\quad - \sum_{i=1}^{k} T(X_1, \ldots, X_{i-1}, \nabla_X X_i, X_{i+1}, \ldots, X_k).\end{aligned}$$

4. Exercise (4 points).

Let (M, g) be a smooth compact Riemannian manifold. For $c \in \mathbb{R}_{>0}$ show that $S_c M := \{X \in TM \mid g(X, X) = c^2\}$ is compact. Then prove that every maximal geodesic of (M, g) is defined on all of $\mathbb{R}$.

Hint: recall what is known for maximally defined solutions of first order ODEs satisfying the Picard-Lindelöf assumptions on an open subsets of $\mathbb{R}^n$.

Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl
Please hand in the exercises until **Tuesday, December 12**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 8

1. Exercise (4 points).
Let $(M_1, g_1), (M_2, g_2)$ be two Riemannian manifolds with the induced Levi-Civita connections ∇^1, ∇^2. We identify (as in Exercise sheet no. 3, Exercise 1)

$$T_{(p,q)}(M_1 \times M_2) \cong T_p M_1 \times T_q M_2$$

and define the product metric $g_1 \oplus g_2$ on $M_1 \times M_2$ by

$$g_1 \oplus g_2 \left((v_1, w_1), (v_2, w_2)\right) = g_1(v_1, v_2) + g_2(w_1, w_2).$$

For vector fields $X_i \in \mathfrak{X}(M_i)$ where $i = 1, 2$ we define $X_1 \oplus X_2 \in \mathfrak{X}(M_1 \times M_2)$ by the formula

$$(X_1 \oplus X_2)|_{(p,q)} = \left(X_1|_p, 0|_q\right) + \left(0|_p, X_2|_q\right).$$

a) Construct a vector $X \in \mathfrak{X}(\mathbb{R}^2)$ that cannot be written as $X = X_1 \oplus X_2$ for vectors fields $X_i \in \mathfrak{X}(\mathbb{R})$.

b) Let $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$ be vector fields on $M_1 \times M_2$. Show that the Levi-Civita connection ∇ of $(M_1 \times M_2, g_1 \oplus g_2)$ satisfies

$$\nabla_Y X = \nabla^1_{X_1} Y_1 + \nabla^2_{X_2} Y_2.$$

c) Let c_1, c_2 be geodesics on M_1 respectively M_2. Conclude, that $c(t) = (c_1(t), c_2(t))$ is a geodesic on $M_1 \times M_2$.

2. Exercise (4 points).
Consider the hyperbolic plane $(\mathfrak{H}, g^{\text{hyp}})$, where

$$\mathfrak{H} = \left\{x + iy \in \mathbb{C} \mid x \in \mathbb{R} \text{ and } y > 0\right\}$$

with metric given by $g^{\text{hyp}}_{x+iy} = \frac{1}{y^2} g^{\text{eucl}}$. Let $r > 0, a \in \mathbb{R}$. Show that the half-circles

$$C_{r,a} = \left\{z \in \mathfrak{H} \mid |z - a| = r\right\}$$

are (up to reparametrisation) geodesics of the hyperbolic plane.
A way to solve this is as follows. First show that one can reduce to the case $(r, a) = (1, 0)$*. Then find a Möbius transformation* $\Psi_A \colon z \mapsto \frac{az+b}{cz+d}$ *where* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2; \mathbb{R})$*, with* $\Psi_A(i) = i$ *and* $\Psi_A(0) = -1$*. Conclude the statement by application of* Ψ_A *to the geodesic* $\gamma(t) = ie^t$*.*

3. Exercise: *Models of the hyperbolic plane* (4 points).
In this Exercise we want to identify three models of the hyperbolic plane.

- The *hyperboloid* model

$$\mathbb{H}^2 = \left\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = -1 \text{ and } z > 0\right\}$$

equipped with the induced metric from $\mathbb{R}^{2,1}$ (as in Sheet no. 7, Exercise 1).

- The *Poincaré half-plane* model

$$\mathfrak{H} = \left\{x + iy \in \mathbb{C} \mid y > 0\right\},$$

with the Riemannian metric $g^{\mathfrak{H}}_{x+iy} = \frac{1}{y^2} g^{\mathrm{eucl}}$.

- The *Poincaré disk* model

$$\mathbb{D} = \left\{x + iy \in \mathbb{C} \mid x^2 + y^2 < 1\right\},$$

equipped with the metric $g^{\mathbb{D}}_{x+iy} = \left(\frac{2}{(1-(x^2+y^2))}\right)^2 g^{\mathrm{eucl}}$.

a) We define a sterographic projection $f\colon \mathbb{H}^2 \to \mathbb{D}$ by the following procedure: Every point $p \in \mathbb{H}^2$ is send to the intersection point of the connecting straight line of p and the point $(0,0,-1)$ with the $x-y$-plane. Show that f is an isometry.

b) Show that the map

$$h : \mathfrak{H} \to \mathbb{D}, \quad z \mapsto \frac{z-i}{z+i}$$

is an isometry.

4. Exercise (4 points).
Let M and N be semi-Riemannian manifolds of the same dimension. Assume that N is connected.

a) Let $f_1, f_2\colon N \to M$ be two isometries. Assume there exists a point $p \in N$ such that $f_1(p) = f_2(p)$ and $d_p f_1 = d_p f_2$ holds. Show that the two isometries coincide.

b) Let $f\colon M \to M$ be an isometry. Show that the fix point set $\mathrm{Fix}(f) = \{p \in M \mid f(p) = p\}$ is a submanifold[1] of M.
Hint: Use the exponential function of M.

[1]A subset $N \subset M^m$ of a smooth manifold M is a submanifold if for every point $p \in N$ there exists a chart $x\colon U \to V$ around the point p such that $x(U \cap N)$ is a submanifold of $\mathbb{R}^m$. Note that this definition does not exclude that different connected components might be of different dimension.

Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl
Please hand in the exercises until **Tuesday, December 19**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 9

1. Exercise (4 points).
Let (M^n, g) be a Riemannian manifold and $x: U \to V$ be a chart of M. Define

$$R^l_{ijk} = dx^l\left(R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}\right)$$

the components of the Riemannian curvature tensor with respect to the chart x. Show that in these coordinate the representation of the curvature tensor in terms of the Christoffel symbols is given by:

$$R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_{m=1}^{n}\left(\Gamma^l_{mi}\Gamma^m_{kj} - \Gamma^l_{mj}\Gamma^m_{ki}\right).$$

2. Exercise (4 points).
Consider the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with induced Riemannian metric $g_{\mathbb{S}^n}$. Let $\{e_i\}_i \subset \mathbb{R}^{n+1}$ be the standard orthonormal basis and define the vector fields $X_i \in \mathfrak{X}(\mathbb{R}^{n+1})$

$$(X_i)\big|_p = e_i - \langle e_i, p\rangle p \text{ for all } p \in \mathbb{R}^{n+1}$$

In this exercise we want to compute the Riemannian curvature tensor of the standard metric of the sphere. We proceed as follows:

a) Show that $X_i\big|_{\mathbb{S}^n} \in \mathfrak{X}(\mathbb{S}^n)$.

b) Recall that the Levi-Civita connection on $\mathbb{S}^n$ is given by $(\nabla_X Y)_{|p} = \pi_p^{\tan}(\partial_X \tilde{Y})$ for $X \in T_pM$ and $Y \in \mathfrak{X}(\mathbb{S}^n)$ with an extension $\tilde{Y} \in \mathfrak{X}(\mathbb{R}^{n+1})$ and $\pi_p^{\tan}$ is the orthogonal projection $\mathbb{R}^{n+1} \to T_p\mathbb{S}^n$. Show:

$$(\nabla_{X_j} X_k)\big|_p = -\langle e_k, p\rangle X_j\big|_p$$

c) Show for $i, j, k \geq 2$: $(R(X_i, X_j)X_k)\big|_{e_1} = -\delta_{ik}e_j + \delta_{jk}e_i$.

d) Show that for all points $p, q \in \mathbb{S}^n$ there exists a $A \in \mathrm{SO}(n+1)$ such that $Ap = q$ holds. Conclude that the full Riemannian curvature of the standard sphere is given by:

$$g_{\mathbb{S}^n}(R(X,Y)Z,T) = g_{\mathbb{S}^n}(Y,Z)g_{\mathbb{S}^n}(X,T) - g_{\mathbb{S}^n}(X,Z)g_{\mathbb{S}^n}(Y,T).$$

3. Exercise (4 points).
Let (M, g) be a Riemannian manifold and $p \in M$ a point in M. Let $\hat{R}$ be a curvature tensor for T_pM, i.e. a tensor $\hat{R} \in T_pM \otimes (T_p^*M)^{\otimes 3}$, which satisfies the following identities:

$$\hat{R}(X_1, X_2, X_3) = -\hat{R}(X_2, X_1, X_3)$$
$$g_p(\hat{R}(X_1, X_2, X_3), X_4) = -g_p(\hat{R}(X_1, X_2, X_4), X_3)$$
$$\hat{R}(X_1, X_2, X_3) + \hat{R}(X_2, X_3, X_1) + \hat{R}(X_3, X_1, X_2) = 0$$

for all $X_1, X_2, X_3, X_4 \in T_pM$. We take a chart $x: U \to V$ with $x(p) = 0$ and construct a Riemannian metric

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{\alpha,\beta} \hat{R}_{i\alpha\beta j} x^\alpha x^\beta$$

on the chart neighborhood U. Show that $R_p = \hat{R}$ holds.

4. Exercise (4 points).
Let (M, g) be a Riemannian manifold and $f: M \to \mathbb{R}$ be a smooth function. We define *gradient vector field* of f by

$$g(\operatorname{grad} f, X) = X(f)$$

for all $X \in \mathfrak{X}(M)$. Moreover we define the *Hessian* of f by

$$\operatorname{Hess}(f)(X, Y) = (\nabla df)(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$.

a) Show that the gradient is a well-defined smooth vector field on M.

b) Let $x: U \to V$ be a chart. Show the local representation of the gradient of f:

$$\operatorname{grad} f|_U = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$$

If (e_i) is a generalized orthonormal basis of T_pM with $g_p(e_i, e_j) = \epsilon_i \delta_{ij}$, then show

$$\operatorname{grad} f|_p = \sum_i \epsilon_i \partial_{e_i} f \cdot e_i$$

c) Show that the Hessian of f is a well-defined $(0,2)$ tensor on M. Does it depend on g?

d) Show that the Hessian is given by $\operatorname{Hess}(f) = \partial_X\big(\partial_Y(f)\big) - (\nabla_X Y)(f)$ and that $\operatorname{Hess}(f)$ is symmetric.

Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl
Please hand in the exercises until **Tuesday, January 9**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 10

1. Exercise: *Polar normal coordinates* (4 points).
Let (M^2, g) be a 2-dimensional Riemannian manifold. Let $p \in M$ be a point and choose an $\epsilon > 0$ such that the exponential map $\exp_p : B_\epsilon(0) \to \exp_p(B_\epsilon(0))$ is a diffeomorphism. Denote by $x = (x^1, x^2)$ the normal coordinates at p and consider the induced *Polar normal coordinates* (r, φ) via the identification $T_pM \cong \mathbb{R}^2$ with euclidean space.

a) Show that we have the following identification of the induced coordinate vector fields:

$$\begin{aligned} \frac{\partial}{\partial r} &= \cos(\varphi)\frac{\partial}{\partial x^1} + \sin(\varphi)\frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial \varphi} &= -r\sin(\varphi)\frac{\partial}{\partial x^1} + r\cos(\varphi)\frac{\partial}{\partial x^2} \end{aligned}$$

b) Determine the coefficients of the metric in Polar normal coordinates $g_{rr}, g_{r\varphi}, g_{\varphi\varphi}$ in terms of the metric g_{ij} with respect to normal coordinates.

c) Let (E_1, E_2) be an orthonormal basis of (T_pM, g_p). Consider the closed curve $\gamma_r(t) = \exp_p(r\cos(t)E_1 + r\sin(t)E_2)$ on M for $t \in [0, 2\pi]$ and a radius $r < \epsilon$. Show that the sectional curvature K_p of (M, g) at p can be computed as follows

$$K_p = \frac{3}{\pi}\lim_{r\to 0}\frac{2\pi r - \mathcal{L}[\gamma_r]}{r^3},$$

where $\mathcal{L}[\gamma_r]$ is the length of the curve γ_r. Can you give a heuristic explanation of this formula? *Hint: Use the Taylor expansion of the metric in normal coordinates and express it then in Polar normal coordinates.*

2. Exercise: *Bianchi identities* (4 points).
Let $\alpha \in \Omega^1(M)$ be a 1-form and $\beta \in \Omega^2(M)$ be a 2-form on a Riemannian manifold (M, g). Let $X_1, X_2, X_3, X_4 \in \mathfrak{X}(M)$ be vector fields on M. Recall the expressions of the Cartan differential:

$$\begin{aligned} d\alpha(X_1, X_2) &= X_1(\alpha(X_2)) - X_2(\alpha(X_1)) - \alpha([X_1, X_2]), \\ d\beta(X_1, X_2, X_3) &= \sum_\sigma X_{\sigma(1)}(\beta(X_{\sigma(2)}, X_{\sigma(3)})) - \beta([X_{\sigma(1)}, X_{\sigma(2)}], X_{\sigma(3)}), \end{aligned}$$

where the sum in the second formula runs over all cyclic permutations of the set $\{1, 2, 3\}$.

a) Show:

$$d\alpha(X_1, X_2) = (\nabla_{X_1}\alpha)(X_2) - (\nabla_{X_2}\alpha)(X_1)$$

b) Use $dd\alpha = 0$ to deduce the first Bianchi identity:

$$R(X_1, X_2)X_3 + R(X_2, X_3)X_1 + R(X_3, X_1)X_2 = 0$$

c) Let $X \in \mathfrak{X}(M)$ be a fixed vector field. Define $\tilde{\alpha}(X_1) = \alpha(\nabla_{X_1} X)$ and deduce, by using $dd\tilde{\alpha} = 0$, the second Bianchi identity:

$$(\nabla_{X_1} R)(X_2, X_3) + (\nabla_{X_2} R)(X_3, X_1) + (\nabla_{X_3} R)(X_1, X_2) = 0$$

3. Exercise (4 points).
Let (M^n, g) be a Riemannian manifold. Denote by R the Riemannian curvature tensor as a $(1,3)$-tensor. Let $X, Y, Z, U, W \in \mathfrak{X}(M)$ be vector fields on M, then define

$$R^{(0,4)}(X, Y, Z, W) = g(R(X, Y)Z, W)$$
$$g(R^{\Lambda^2}(X \wedge Y), Z \wedge W) = R(X, Y, Z, W)$$

the associated $(0,4)$-tensor and curvature endomorphism.

a) Let $\{e_i\}_i \subset T_pM$ be an orthonormal basis of g. Show that by

$$g_p(e_i \wedge e_j, e_k \wedge e_l) = \delta_{ik}\delta_{jl}$$

for $i < j$ and $k < l$ we obtain a non-degenerated bilinearform on T_pM, which depends smoothly on p.

b) Show that $R^{(0,4)}$ is a well-defined $(0,4)$-tensor on M and R^{Λ^2} is a well-defined map $\Lambda^2 T_pM \to \Lambda^2 T_pM$, which depends smoothly on p.

c) Show that we have the following identities:

$$(\nabla_X R^{(0,4)})(Y, Z, U, W) = -(\nabla_X R^{(0,4)})(Z, Y, U, W)$$
$$= (\nabla_X R^{(0,4)})(U, W, Y, Z) = -(\nabla_X R^{(0,4)})(Y, Z, W, U)$$

d) Let $T \in \Gamma(T^{(0,s)}M)$ be a $(0,s)$-tensor for $s \geq 1$. We define the *divergence* of T by

$$\mathsf{div}(T)(X_1, \ldots, X_{s-1}) := \sum_{j=1}^{n} (\nabla_{e_j} T)(e_j, X_1, \ldots, X_{s-1}),$$

where $\{e_j\}_j$ is an orthonormal basis of T_pM and $X_1, \ldots, X_{s-1} \in T_pM$. Show:

$$\mathsf{div}(\mathrm{ric}) = \frac{1}{2} d\,\mathrm{scal}\,.$$

Hint: Use the second Bianchi identity for the Riemannian curvature tensor.

4. Exercise: *Schur's Lemma* (4 points).
Let (M^n, g) be a Riemannian manifold.

a) Assume $n \geq 2$ and the sectional curvature K_p only depends on the point p. Then Riemannian curvature tensor is of the form

$$g(R(X, Y)Z, W) = \kappa \cdot (g(X, Z)g(Y, W) - g(Y, Z)g(X, W))$$

where $\kappa \colon M \to \mathbb{R}$ is a smooth function.

b) Assume $n \geq 3$ and the Riemannian curvature tensor is of the form above. Show that $\mathrm{ric} = (n-1)\kappa g$ holds and that in this case that the function κ is locally constant. *Hint: Use Exercise* 3, *d*).

Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl
Please hand in the exercises until **Tuesday, January 16**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 11

1. Exercise (4 points).
Let (M,g) be a semi-Riemannian manifold and $N \subset M$ be an open subset. Assume that N is geodesically complete[1] and M is connected. Show that $N = M$ holds. *Hint: Consider a point in the boundary* $\bar{N} \setminus N$.

2. Exercise (4 points).
Let $N \subset M$ be a semi-Riemannian submanifold of the semi-Riemannian manifold (M,g). We say that N is *totally geodesic* if the second fundamental form $\vec{\mathrm{II}} \equiv 0$ vanishes.

a) Show that N is totally geodesic iff every geodesic of N is also a geodesic of M.

b) Assume now that N is geodesically complete. Show that N is totally geodesic iff every geodesic $\gamma\colon I \to M$, of M with $\dot{\gamma}(0) \in TN$ is contained in N.

c) Do we need the assumption of geodesic completeness in part b) to conclude the statement?

3. Exercise (4 points).
Let $(\bar{M},\bar{g})$ be a semi-Riemannian manifold and M be submanifold of dimension $n = \dim(M) = \dim(\bar{M}) - 1$. Assume that there exists a map into the normal bundle $\nu\colon M \to \mathcal{N}M$, such that $g(\nu,\nu) = \epsilon \in \{-1,+1\}$ holds. Denote by g the induced Riemannian metric on M.

a) Show that there exists a unique bundle map $W \in \Gamma(\mathrm{End}(TM))$ with the property

$$g(W(X),Y) = \bar{g}(\vec{\mathrm{II}}(X,Y),\nu)$$

for all $X,Y \in T_pM$ and $p \in M$. In particular, the endomorphism $W|_p\colon T_pM \to T_pM$ is self-adjoint. We call W the *Weingarten map* of the embedding $(M,g) \hookrightarrow (\bar{M},\bar{g})$.

b) Show that $W(X) = -\bar{\nabla}_X \nu$ holds for all $X \in TM$.

c) Assume that $\bar{M}$ is Riemannian and $n = \dim(M) \geq 3$. Moreover the metric on $\bar{M}$ is assumed to be flat, i.e. $\bar{R} \equiv 0$. Show that for any point $p \in M$ there is a plane $E \subset T_pM$ with $K(E) \geq 0$. *Hint: Consider planes* $E = \mathrm{span}(\xi_i,\xi_j)$ *which are spanned by an orthonormal basis* $\xi_1,\ldots,\xi_n$ *of eigenvectors of* W *and use the Gauß formula.*

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[1] A semi-Riemannian manifold N is *geodesically complete* if the exponential map is defined on the full tangent bundle TN

4. Exercise (4 points).
Let $(\bar{M}, \bar{g})$ be a flat semi-Riemannian manifold and M be a semi-Riemannian submanifold of $\bar{M}$ with dimension m and induced metric g. Let $(b_1, \dots, b_m)$ be a generalized orthonormal basis of T_pM with the condition $g(b_i, b_j) = \delta_{ij}\,\varepsilon_i$, $\varepsilon_i \in \{-1, 1\}$. We define the *mean curvature vector field* by $\vec{H}_p := \sum_{i=1}^{m} \varepsilon_i \,\vec{\mathrm{II}}(b_i, b_i)$.

a) Show that $\vec{H}_p$ is well-defined.

b) Show that

$$\mathrm{Ric}(X, Y) = \bar{g}(\vec{H}_p, \vec{\mathrm{II}}(X, Y)) - \sum_{i=1}^{m} \varepsilon_i \,\bar{g}(\vec{\mathrm{II}}(b_i, X), \vec{\mathrm{II}}(b_i, Y)).$$

holds for all $X, Y \in T_pM$

c) Let M be of dimension $m-1$ and assume that there exists a map into the normal bundle $\nu\colon M \to \mathcal{N}M$, such that $g(\nu, \nu) = \epsilon \in \{\pm 1\}$ holds with associated Weingarten map W (defined in Exercise 3). Show that:

$$\bar{g}(\nu, \nu) \cdot \mathrm{scal} = (\mathrm{Tr}\, W)^2 - \mathrm{Tr}(W^2).$$

Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl
Please hand in the exercises until **Tuesday, January 23**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 12

1. Exercise (4 points).
Let (M^2, g) be a two-dimensional Riemannian submanifold of $\mathbb{R}^3$. We call M a *minimal surface* if the mean curvature of M in $\mathbb{R}^3$ vanishes.

a) Show that a minimal surface has non-positive sectional curvature, and if the sectional curvature is 0 in $p \in M$, then the fundamental form vanishes in p.

b) Consider the *catenoid*

$$\Phi_1 : \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x, y) \mapsto \begin{pmatrix} \alpha \cosh(x) \cos(y) \\ \alpha \cosh(x) \sin(y) \\ \sinh(x) \end{pmatrix}$$

and the *helicoid*

$$\Phi_2 : \mathbb{R}^2 \to \mathbb{R}^3$$
$$(x, y) \mapsto \begin{pmatrix} x \cos(y) \\ x \sin(y) \\ \beta y \end{pmatrix}$$

with constants $\alpha, \beta \in \mathbb{R}$. Compute the induced metrics g_1, g_2 on $\mathbb{R}^2$ and the Weingarten maps. Show that the catenoid and the helicoid are minimal surfaces in $\mathbb{R}^3$.

c) Compute the sectional curvatures of both surfaces. Does there exists an isometry $\phi : (\mathbb{R}^2, g_1) \to (\mathbb{R}^2, g_2)$?

d) Show that there does not exists an isometry $\bar{\phi} : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\bar{\phi}(\mathrm{image}(\Phi_1)) = \mathrm{image}(\Phi_2)$ holds.

2. Exercise (4 points).
Let (M^n, g) be a Riemannian manifold with non-positive sectional curvature, i.e. $K \leq 0$. We denote by J a Jacobi field along a geodesic c of (M, g).

a) Show that $g(J, \frac{\nabla^2}{dt^2} J)$ is a non-negative function.

b) Show that $\frac{d^2}{dt^2}(g(J, J))$ is a non-negative function.

c) Conclude from the previous statements that the Jacobi field vanishes identically or has at most one point where it vanishes.

3. Exercise (4 points).
Let (M,g) be a semi-Riemannian manifold and J be a Jacobi field along a geodesic $c: I = [a,b] \to M$. Show that there exists a geodesic variation $c_\bullet: (-\epsilon,\epsilon) \times I \to M$ of c such that $J = \frac{d}{ds}|_{s=0} c_s$ holds.
Hint: For some $t_0 \in [a,b]$ choose a curve $\gamma : (-\epsilon,\epsilon) \to M$ with $\gamma(0) = c(t_0)$ and $\dot\gamma(0) = J(t_0)$. Find a vector field X along γ such that $(s,t) \mapsto c_s(t) = \exp_{\gamma(s)}(tX(s))$ is a suitable geodesic variation.

4. Exercise (4 points).
Let (M,g) be a semi-Riemannian manifold.

a) Recall that we denote the parallel transport along a curve γ by $\mathcal{P}_\gamma$. Let $F: \mathbb{R}^2 \to M$ be a smooth map and denote by γ_t the curve in M which is given by

$$\gamma_t(s) = \begin{cases} F(4st, 0) & s \in [0, \frac{1}{4}] \\ F(t, t(4s-1)) & s \in [\frac{1}{4}, \frac{1}{2}] \\ F(t(3-4s), t) & s \in [\frac{1}{2}, \frac{3}{4}] \\ F(0, t(4-4s)) & s \in [\frac{3}{4}, 1], \end{cases}$$

i.e. the piecewise smooth curve which gives the image of the closed polygonal chain with corner points $(0,0), (t,0), (t,t)$ and $(0,t)$. Show that

$$\lim_{t\to 0} \frac{\mathcal{P}_{\gamma_t} v - v}{t^2} = R\left(\frac{\partial F}{\partial x_2}(0), \frac{\partial F}{\partial x_2}(0)\right) v$$

holds for all $v \in T_{F(0,0)}M$.
Hint: Use the following statement from the lecture (Lemma V.4.2): Let $\alpha: \mathbb{R}^2 \to M$ be a smooth map and X a vector field along α such that $\frac{\nabla}{\partial x} X = \frac{\nabla}{\partial y} X$ holds, then we have

$$\frac{\nabla}{\partial x}\frac{\nabla}{\partial x} X - \frac{\nabla}{\partial y}\frac{\nabla}{\partial y} X = R\left(\frac{\partial \alpha}{\partial x}, \frac{\partial \alpha}{\partial y}\right) X.$$

b) If (M,g) is flat, then for every point $p \in M$ and vector $v \in T_pM$, there exists an open neighbourhood of p given by $U \subset M$ and a section $X: U \to TM$ of the tangent bundle TM, which is parallel, i.e. $\nabla X = 0$ on U, and satisfies $X_p = v$. Construct a counterexample in the non-flat case for the previous statement.

Differential Geometry I: Exercises

University of Regensburg, Winter Term 2023/24
Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl
Please hand in the exercises until **Tuesday, January 30**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 13

1. Exercise (4 points).
Let M be a compact surface (without boundary) in $\mathbb{R}^3$. Let $\overline{B}_r(0)$ be the closed ball of radius r around 0 in $\mathbb{R}^3$, and let $S_r(0) = \partial \overline{B}_r(0)$ be its boundary.

a) Show that the infimum $R := \inf\{r > 0 \,|\, M \subset \overline{B}_r(0)\} > 0$ is attained, and conclude that $M \cap S_R(0)$ is not empty.

b) Show that T_pM is the orthogonal complement of p for any $p \in M \cap S_R(0)$. Show for any such $p \in M$ that the symmetric bilinear form

$$T_pM \times T_pM \to \mathbb{R}, \quad (X,Y) \mapsto \left\langle \tfrac{1}{R}p, \vec{\mathrm{II}}(X,Y)\right\rangle$$

is negative definit.

c) Are there compact minimal surfaces M in $\mathbb{R}^3$? Justify your answer.

2. Exercise (4 points).
Let (M,g) be a connected, non-compact, geodesically complete Riemannian manifold and $p \in M$ be a point. You may use the facts that under these conditions (M,d) is a complete metric space and that for any $p,q \in M$ there is a shortest curve from p to q.

a) Show the existence of a sequence points $\{p_i\}_{i\in\mathbb{N}}$ in M with $d(p,p_i) \to \infty$ for $i \to \infty$.

b) Conclude the existence of a geodesic ray[1] $\gamma\colon [0,\infty) \to M$ with $\gamma(0) = p$.
Hint: Consider a length minimizing geodesic $\gamma_i\colon [0,l_i] \to M$ with $\gamma_i(0) = p$ and $\gamma_i(l_i) = p_i$. Use the fact that $\|\dot\gamma_i(0)\| = 1$ to conclude that there exists convergent subsequence $\dot\gamma_{i_j}(0) \to X \in T_pM$. Consider then $\gamma(t) = \exp_p(tX)$ and show $d(p,\gamma(t)) = t$.

3. Exercise (4 points).
Let (M,g) be a connected, geodesically complete Riemannian manifold and $N \subset M$ be a closed submanifold.[2] We fix a point $q \in M \setminus N$. We denote by $d(x,N) := \inf\{d(x,y) \,|\, y \in N\}$ the minimal distance from x to the submanifold N.

a) Show that there exists a point $p \in N$ with $d(q,p) = d(q,N)$. Do we need the assumption that N is closed?

b) Show the existence of a geodesic γ, which connects p and q with length given by $\mathcal{L}(\gamma) = d(q,p)$.

c) Conclude with the first variation of the energy that the curve γ hits N in an orthogonal way.

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[1] A geodesic ray $\gamma\colon [0,\infty) \to M$ is a geodesic such that for all compact subsets $K \subset M$ there exists a time $T > 0$ such that $\gamma(T) \notin K$ holds.

[2] You may use the facts that under these conditions (M,d) is a complete metric space and that for any $p,q \in M$ there is a shortest curve from p to q.

4. Exercise (4 points).
Let M be a smooth manifold and G be a group equipped with the discrete topology. Moreover we have a continuous group action

$$\begin{aligned} R: M \times G &\to M \\ (p, g) &\mapsto R(p, g), \end{aligned}$$

i.e. R satisfies $R(p, gh) = R(R(p, g), h)$ for all $p \in M$ and $g, h \in G$. We denote by $p \cdot G := \{R(p, g) \mid g \in G\}$ the orbit of p along the group action and we denote by $M/G := \{p \cdot G \mid p \in M\}$ the *quotient space* of the group action. The *canonical projection* $\pi: M \to M/G, p \mapsto \pi(p) = p \cdot G$ induces a topology on the quotient M/G, i.e. a subset $U \subset M/G$ is open iff $\pi^{-1}(U) \subset M$ is open.

a) Show that the right multiplication maps $R_g: M \to M, p \mapsto R(p, g)$ is a homeomorphism for any $g \in G$. Are these maps also diffeomorphisms?

Now we assume that the group action R is free and properly discontinuous. Here we refer to an action R as free if for any $g \in G \setminus \{e\}$ the right multiplication maps R_g has no fixed point. An action R is properly discontinuous if for all points $p, q \in M$ there exist open neighbourhoods U_p, V_q of p respectively q such that $R_g(U_p) \cap V_q = \emptyset$ holds for all $g \in G$ with the condition $R(p, g) \neq q$.

b) Show that the quotient space M/G is Hausdorff.

c) Show that the canonical projection $\pi: M \to M/G$ is a covering map, i.e. for all points $P \in M/G$ there exists an open neighbourhood U of P and a homeomorphism $\Phi_U: \pi^{-1}(U) \to U \times G$ such that $\Phi \circ \mathrm{pr}_1 = \pi$ holds.

d) (Bonus part) Assume additionally that R_g is smooth for any $g \in G$. Show then that the quotient space M/G is a smooth manifold and the canonical projection is a local diffeomorphism.

Prof. Dr. Bernd Ammann, Julian Seipel, Roman Schießl

Please hand in the exercises until **Tuesday, February 6**
12 noon in the letterbox of your group (no. 15 or 16)

Exercise Sheet no. 14

1. Exercise (4 points).
Let $\varphi\colon (M,g) \to (N,h)$ be a smooth map between connected manifolds and $g = \varphi^* h$ is the pullback of the metric h.

a) If φ is a covering map, then show that (M,g) is complete iff (N,h) is complete.

b) Assume that φ is a local diffeomorphism and an isometry. Show that if (M,g) is complete, then the map φ is a covering map.

2. Exercise (4 points).
Let $(M^{n\geq 2}, g)$ be connected, complete Riemannian manifold with constant sectional curvature. Assume moreover that M is simply-connected. Show

$$(M,g) \text{ is isometric to } \begin{cases} \mathbb{H}^n & \text{if } K = -1, \\ \mathbb{R}^n & \text{if } K = 0, \\ \mathbb{S}^n & \text{if } K = 1. \end{cases}$$

3. Exercise (4 points).
Let $\varphi\colon (M,g) \to (N,h)$ be a surjective submersion between connected complete Riemannian manifolds. We call φ a *Riemannian submersion* if the map $d_p\varphi$ induces an isomorphism $H_pM := (\ker(d_p\varphi))^\perp \to T_{\varphi(p)}N$ for each $p \in M$. We call $HM := \bigcup_{p\in M} H_pM \subset TM$ the horizontal subbundle and its elements *horizontal*.

a) Let $\gamma : I \to N$ be a smooth curve, I some interval. Show that there exists a horizontal lift $\tilde{\gamma} : I \to M$, i.e. a curve $\tilde{\gamma}$ satisfying $\dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}(t)}M$ and $\varphi \circ \tilde{\gamma} = \gamma$. Also show for any curve $\tau : [a,b] \to N$ that $\mathcal{L}(\varphi \circ \tau) \leq \mathcal{L}(\tau)$.

b) Show: if γ is a geodesic, then its horizontal lift $\tilde{\gamma}$ is also a geodesic. *Hint: use the fact that γ locally minimizes length to show that $\tilde{\gamma}$ also minimizes length locally.*

c) Show: if a horizontal curve $\tau : I \to M$ is geodesic, then $\varphi \circ \tau : I \to N$ is also a geodesic.

d) Let γ be a geodesic in M. Show that if $\dot{\gamma}(0)$ lies in $H_{\gamma(t)}M$ then we have $\dot{\gamma}(t) \in HM$ for all $t \in I$.

4. Exercise (4 points).
Let (M^n, g) be a Riemannian manifold. We assume that (M,g) is *locally symmetric*, i.e. $\nabla R = 0$ holds. In this exercise we want to show that this condition is equivalent to the existence of a local isometry $\sigma_p : U \to \sigma(U)$ with $\sigma(p) = p$ and $d_p\sigma = -\mathrm{id}_{T_pM}$, defined on open neighbourhood $U \subset M$ of p.

a) Let $\epsilon > 0$ small enough such that the exponential function is a diffeomorphism onto its image, i.e. $\exp_p: B_\epsilon(0) \overset{\simeq}{\to} \exp_p(B_\epsilon(0)) = B_\epsilon(p)$. We define the map

$$\begin{aligned} \sigma_p: B_\epsilon(p) &\to B_\epsilon(p) \\ \gamma(t) &\mapsto \gamma(-t), \end{aligned}$$

where we use that each point in $B_\epsilon(p)$ can be represented by a geodesic emanating from p. Show that $\sigma_p = \exp_p \circ (-\operatorname{id}_{T_pM}) \circ \exp_p^{-1}$ holds.

b) Let $v \in B_\epsilon(0)$ and $q = \exp_p(v)$. Moreover let $\gamma(t) = \exp_p(tv)$ and $\bar{\gamma}(t) = \gamma(-t)$ be curves in M. We consider the map

$$\begin{aligned} F_t : T_{\gamma(t)}M &\to T_{\bar{\gamma}(t)}M \\ w &\mapsto \mathcal{P}^{\bar{\gamma}}_{0,t} \circ (-\operatorname{id}_{T_pM}) \circ \mathcal{P}^{\gamma}_{t,0}(w), \end{aligned}$$

where $\mathcal{P}^c_{a,b}: T_{c(a)}M \to T_{c(b)}M$ denotes the parallel transport along the curve $c: I \to M$ with $a, b \in I$. Show that for each Jacobi field $J(t)$ along γ, the field $\bar{J}(t) = F_t(J(t))$ is a Jacobi field along $\bar{\gamma}$. Conclude from the previous statement that the map $\sigma_p: B_\epsilon(p) \to B_\epsilon(p)$ is an isometry.

c) Let $\gamma: (-\epsilon, \epsilon) \to M$ be a geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Moreover assume that (M, g) is not nessecarily locally symmetric and all the maps σ_p from part a) are isometries. Show for a parallel frame $(e_1(t), \dots, e_n(t))$ along γ we have

$$g_{\gamma(t)}(R(e_i(t), e_j(t)e_k(t)), e_l(t)) = g_{\gamma(-t)}(R(e_i(-t), e_j(-t)e_k(-t)), e_l(-t))$$

and conclude that (M, g) is locally symmetric.