Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
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Please hand in the exercises until **Monday, July 10th** in the lecture

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Exercise Sheet no. 11

Exercise 1 (4 points).
Let G be a smooth manifold and a group. Assume that the group multiplication $m: G \times G \to G$ is a smooth map. Show that the inversion $\iota: G \to G$ is also smooth.
Hint: Use the equation $m(\iota(g), g) = e$ for all $g \in G$ and e the neutral element of the group.

Exercise 2 (4 points).
Let $S_1, S_2 \in \mathbb{R}^{n\times n}$ be symmetric matrices. Define for A and B in the Lie algebra $\mathfrak{so}(n) := \{M \in \mathbb{R}^{n\times n} \mid M^T = -M\}$ given by

$$\langle A,B\rangle_{S_1,S_2} := \operatorname{tr}\left(S_2 A S_1 B^T\right). \tag{1}$$

a) Let S_1 and S_2 be positive definit. Show that $\langle \cdot,\cdot \rangle_{S_1,S_2}$ is a scalar product on $\mathfrak{so}(n)$.

b) Show that for any scalar product $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ on $\mathfrak{so}(3)$, there exists a unique symmetric matrix $S_3 \in \mathbb{R}^{n\times n}$ with $\langle \cdot,\cdot \rangle_{S_3,\mathrm{id}} = \langle\!\langle \cdot,\cdot \rangle\!\rangle$. Is S_3 always positive definit.

c) Prove the existence of scalar products on $\mathfrak{so}(4)$ which are not of the form as in eq. (1).

d) Show that every scalar product on $\mathfrak{so}(n)$ is a sum of scalar products as in eq. (1) with positive definit S_1 and S_2.

Exercise 3 (4 points).
Let G be a Lie group. We consider the adjoint action of G on its Lie algebra:

$$\mathrm{Ad}: G \to \mathrm{GL}(\mathfrak{g}), g \mapsto (X \mapsto \mathrm{d}_e l_g \circ \mathrm{d}_e r_{g^{-1}}(X))$$

where $l, r: G \to \mathrm{Aut}(G)$ are the left and right multiplication of the group G. Let $\langle \cdot,\cdot \rangle$ be a scalar product on $\mathfrak{g}$. We say that $\langle \cdot,\cdot \rangle$ is Ad-invariant if $\mathrm{Ad}_g^* \langle \cdot,\cdot \rangle = \langle \cdot,\cdot \rangle$ holds for any $g \in G$. A tensor (field) on G is called bi-invariant, if it is both left- and right-invariant.

a) Show: a scalar product $\langle \cdot,\cdot \rangle$ on $\mathfrak{g}$ can be extended to a bi-invariant Riemannian metric, if and only if $\langle \cdot,\cdot \rangle$ is Ad-invariant.

b) Assume that we know $\mathrm{d}_e(\mathrm{Ad})(X) = \mathrm{ad}_X = [X, \cdot]$ for all $X \in T_e g = \mathfrak{g}$. Show: if $\langle \cdot,\cdot \rangle$ is Ad-invariant, then ad_X is skew-symmetric w.r.t. $\langle \cdot,\cdot \rangle$. Is the converse true as well? Or is it true under additional assumptions?

c) Let γ be a left-invariant Riemannian metric extending the scalar product $\langle \cdot,\cdot \rangle$ on $\mathfrak{g}$. Show that ad_X is skew-symmetric w.r.t. $\langle \cdot,\cdot \rangle$, iff the Levi-Civita connection for γ is given by $\nabla^\gamma_X Y = \frac{1}{2}[X, Y]$ for all $X, Y \in \mathfrak{g}$. *Hint: Use the Koszul formula.*

d) Show that the induced Riemannian exponential function and the Lie exponential on G coincide for a biinvariant metric γ.

e) *Bonus exercise:* Let G be a connected Lie group with a left invariant metric γ. Show that if the induced Riemannian exponential map and the Lie exponential map coincide, then γ is biinvariant.

Exercise 4: *Coadjoint orbit* (4 points).
Let G be a Lie group and $\mathfrak{g}$ its Lie algebra and $\xi \in \mathfrak{g}^*$ be an element in the dual. We have the adjoint action $\mathrm{Ad}: G \to \mathrm{GL}(\mathfrak{g})$ and the induced coadjoint action Ad^* given by $(\mathrm{Ad}_g^* \xi)(X) = \xi(\mathrm{Ad}_g X)$ for $X \in \mathfrak{g}, \xi \in \mathfrak{g}^*$ and $g \in G$. Note that the coadjoint action Ad^* is a right action, i.e. $\mathrm{Ad}_{gh}^* = \mathrm{Ad}_h^* \circ \mathrm{Ad}_g^*$. Similarly we define $\mathrm{ad}_X^* \xi \in \mathfrak{g}$ by

$$(\mathrm{ad}_X^* \xi)(Y) \coloneqq \xi(\mathrm{ad}_X(Y)) = \xi([X,Y]) \quad \forall Y \in \mathfrak{g} \,.$$

For a fixed covector $\mu \in \mathfrak{g}^*$ we define the *coadjoint orbit* $\mathcal{O}_\mu \coloneqq \{\mathrm{Ad}_g^*(\mu) \mid g \in G\}$.

a) Show that the coadjoint orbit $\mathcal{O}_\mu$ is submanifold of $\mathfrak{g}^*$, whose tangent space at ν is $\{\mathrm{ad}_X^* \nu \mid X \in \mathfrak{g}\}$.
   If helpful, you may use without proof, that any closed subgroup H of a Lie group G is a submanifold, and then G/H carries a unique manifold structure, such that the projection $G \to G/H$ is a submersion.
   Show that Ad^* defines a smooth and transitive action of G on $\mathcal{O}_\mu$.

b) For $\nu \in \mathcal{O}_\mu$ and $X, Y \in T_\nu \mathcal{O}_\mu$ we define

$$\omega_\nu : T_\nu \mathcal{O}_\mu \times T_\nu \mathcal{O}_\mu \to \mathbb{R}, \quad \omega_\nu(\mathrm{ad}_X^* \nu, \mathrm{ad}_Y^* \nu) \coloneqq \nu([X,Y]) \,.$$

   Show that ω_ν is an alternating, non-degenerate bilinear map.

c) Show that this 2-form ω is a G-invariant symplectic form on the coadjoint orbit $\mathcal{O}_\mu$. In fact, show that ω is closed and that for all $g \in G$ we have $\left(\mathrm{Ad}_g^*\right)^* \omega = \omega$.