Exercise Sheet no. 10

Exercise 1 (4 points).
Let (M, ω, g, J) be a manifold equipped with a non-degenerated 2-form ω, a Riemannian metric g and an almost complex structure J. Assume that ω and J are compatible with $g = \omega(\cdot, J\cdot)$. Show that if J is g-parallel, i.e. $\nabla^g J = 0$, then J is integrable and ω is parallel and moreover ω is closed.

Exercise 2 (4 points).
Let $P_1, \dots, P_k$ be homogeneous Polynomials in $(n+1)$-variables. Assume that for every point $z \in \mathbb{C}^{n+1} \setminus \{0\}$ with $P_1(z) = \dots = P_k(z) = 0$ the differentials

$$\mathrm{d}_z P_1, \dots, \mathrm{d}_z P_k$$

are linear independent. Show that subset

$$\bigcap_{i=1}^{k} P_i^{-1}(\{0\}) \subset \mathbb{C}P^n$$

is a complex submanifold.

Exercise 3: *Segre embedding* (4 points).
Let V, W be finite dimensional complex vector spaces. The map

$$\iota_{V,W} : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W)$$
$$([v],[w]) \mapsto [v \otimes w]$$

is called the *Segre embedding*, where we denote the projectivization of V by $\mathbb{P}(V) = V \setminus \{0\}/\sim$ with the equivalence relation $\sim$ given by: Let $v, w \in V \setminus \{0\}$ be equivalent $v \sim w$ if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $v = \lambda w$. Show:

a) Let M be a complex manifold and $N \subset M$ a real submanifold and assume that the integrable complex structure J^M of M preserves the tangent bundle of N, i.e. $J^M(TN) \subset TN$, then N is a complex submanifold of M.

b) The map ι is an embedding and the image is a complex submanifold. What is the codimension of the image?

Exercise 4 (4 points).
Let M^{2n} be a complex manifold with real dimension $2n$. Recall that the complexified tangent bundle $T_{\mathbb{C}}M$ splits into the $\pm i$-Eigensubbundles of the integrable complex structure J of M, i.e. $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$. Write $\Lambda^{1,0} = T^{1,0}M$ and $\Lambda^{0,1} = T^{0,1}M$. We define $\Lambda^{p,0} := \bigwedge_{i=1}^{p} \Lambda^{1,0}$ and respectively $\Lambda^{0,q} := \bigwedge_{i=1}^{q} \Lambda^{0,1}$. We have the map

$$\iota : \Lambda^{p,0} \otimes \Lambda^{0,q} \to \Lambda^k_{\mathbb{C}} := \Lambda^k \otimes_{\mathbb{R}} \mathbb{C}$$
$$\alpha \otimes \beta \to \alpha \wedge \beta$$

and set $\Lambda^{p,q} := \mathrm{image}(\iota)$.

a) Show that $T^*M \otimes_{\mathbb{R}} \mathbb{C} \cong (TM \otimes_{\mathbb{R}} \mathbb{C})^*$ holds.

b) Let $(\varphi: U \subset M \to V \subset \mathbb{C}^n, z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ be a complex chart of M such that $T^*_{\mathbb{C}}M$ is trivialized over U. Show that $\mathrm{d}z_1, \ldots \mathrm{d}z_n, \mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n$ is a basis of $T^*_{\mathbb{C}}U$.

c) Construct an isomorphism $\bigoplus_{p+q=k} \Lambda^{p,q} \cong \Lambda^k_{\mathbb{C}}$.