Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
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Please hand in the exercises until **Monday, June 26th** in the lecture

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Exercise Sheet no. 9

Exercise 1 (4 points).
Let (M, ω) be a symplectic manifold and recall that the Poisson bracket is given by $\{f, g\} \coloneqq \omega(\operatorname{sgrad} f, \operatorname{sgrad} g)$. Show that in Darboux coordinates $(U, (q_i, p_i))$, i.e. $\omega_{|U} = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$ holds, the Poisson bracket $\{f, g\}$ can be written as

$$\sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

Exercise 2 (4 points).
We consider the following complex structures on $\mathbb{C}^2$

$$I_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and define $J \coloneqq \sum_{k=1}^3 f_k I_k$ for constants $f_k \in \mathbb{R}$ with the condition $f_1^2 + f_2^2 + f_3^2 = 1$.

a) For which $(f_k)_k$ is the structure J a compatible complex structure for the symplectic form $\omega = -\mathfrak{Im}(\langle \cdot, \cdot \rangle_{\mathbb{C}^2})$?

b) For which $(f_k)_k$ is the structure J a compatible complex structure for the symplectic form $\omega = -\mathfrak{Im}(\langle I_1 \cdot, \cdot \rangle_{\mathbb{C}^2})$?

Bonus: Consider the almost complex structure $J \coloneqq \sum_{k=1}^3 f_k I_k$ on the symplectic manifold $(\mathbb{C}^2, \omega_{\text{std}})$ with functions $f_k \colon \mathbb{C}^2 \to \mathbb{R}$ which satisfy the constraint $f_1^2 + f_2^2 + f_3^2 = 1$ on $\mathbb{C}^2$. Show that J is an integrable almost complex structure iff f_k is constant for all k.

Exercise 3 (4 points).
Show that the space of all complex structure on $\mathbb{C}^n$, which are compatible with the standard scalar product, is given by the space

$$\mathrm{O}(2n)/\mathrm{U}(n).$$

Show moreover, that for the case $n = 2$ this space can be identified with

$$S^2 \sqcup S^2.$$

Exercise 4 (4 points).
Consider the map

$$f \colon \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^4 \setminus \{0\}$$
$$x \to \frac{x}{\|x\|^2}$$

and a complex linear structure $J \in \operatorname{End}(\mathbb{R}^4)$.

a) Show that the pullback $J_f := f^*J$ is a complex structure on $\mathbb{R}^4 \setminus \{0\}$. Is there an extension of J_f to all of $\mathbb{R}^4$?

b) Let ∂_r the radial vector on $\mathbb{R}^4 \setminus \{0\}$. We decompose the space $\alpha: \mathbb{R}^4 \setminus \{0\} \cong S^3 \times \mathbb{R}_{>0}, x \mapsto (\frac{x}{\|x\|}, \|x\|)$ and thus also the tangent bundle as

$$\mathrm{d}\alpha: T_p\mathbb{R}^4 \setminus \{0\} \to T_{p/\|p\|}S^3 \oplus T_{\|p\|}\mathbb{R}_{>0}.$$

Show there exists a vector field $X \in \Gamma(TS^3)$, s.t. $(\partial_r, J\partial_r, X, JX)$ is an orthonormal frame of $T\mathbb{R}^4 \setminus \{0\}_{|S^3}$. Show moreover that $\mathrm{d}f$ can be decomposed as

$$\mathrm{d}\alpha \circ \mathrm{d}f \circ \mathrm{d}\alpha^{-1} = \mathrm{id}_{T_{\varphi p}S^3} \oplus -\mathrm{id}_{T_1\mathbb{R}_{>0}}$$

for a point $p \in S^3$.

c) Show that the commutator of the $[J, J_f] = 0$ vanishes. *Bonus: Can you generalize the argument for arbitrary linear complex structures* $J_1, J_2 \in \mathrm{End}(\mathbb{R}^4)$, *which induce different orientations on* $\mathbb{R}^4$*?*

d) We consider the restricted map J_f on the bundle $\pi^{S^3}: \mathbb{R}^4 \times S^3 \to S^3$. Show that for every point $p \in S^3$, the map J_f is constant along a Hopf circle $C_p := S^3 \cap \mathrm{span}\{\partial_r, J_f\partial_r\}_p$.

e) Let $H: S^3 \to \mathbb{C}P^1$ be the quotient map. Let $[p] = L \in \mathbb{C}P^1$ be a complex line and define the following map:

$$\begin{aligned} \varphi: \mathbb{C}P^1 &\to S^2 \\ [p] &\mapsto J(C_p), \end{aligned}$$

where $J(C_p)$ is induced complex structure on $\mathbb{R}^4$ given by $(\partial_r, J_f\partial_r, X, J_fX)$. Show that φ is a diffeomorphism.