Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
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Please hand in the exercises until **Monday, June 19th** in the lecture

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Exercise Sheet no. 8

Exercise 1: *conformal symplectic maps* (4 points).
Let (M_1, ω_1) and (M_2, ω_2) be a symplectic manifolds. A smooth map $f\colon M_1 \to M_2$ is called a *conformal symplectic map* if there exists a smooth function $h\colon M_1 \to \mathbb{R}$ with

$$f^*\omega_2 = h \cdot \omega_1 .$$

a) Show that if the dimension of M_1 is strictly bigger than 2, then the factor h is locally constant.

b) Find conformal symplectic maps for $M_1 = M_2 = \mathbb{R}^{2n}$ and $n > 1$, s.t. the conformal factor h is 0 or ± 1.

c) Find a conformal symplectic map of $M_1 = M_2 = S^2$ s.t. the conformal factor is non-constant.

d) *Bonus:* Let $h \in C^\infty(S^2)$ be given. Can you find a conformal symplectic map $f\colon S^2 \to S^2$ with conformal factor h?

Exercise 2 (4 points).
We consider the following map

$$\begin{aligned} \iota\colon (B_1(0), \omega_{\mathrm{std}}) &\to (\mathbb{C}P^1, \omega_{\mathrm{FS}}) \\ z &\mapsto [z : \sqrt{1 - |z|^2}], \end{aligned}$$

where $B_1(0) \subset \mathbb{C}$ is the standard open ball of radius 1 in the complex plane and the induced standard symplectic form of $\mathbb{C}$. The symplectic form ω_{FS} is the Fubini-Study form introduced on Exercise sheet 7, Ex. 4.
Show that the image of ι is open and dense, and that ι is a symplectomorphism on its image.

Exercise 3: *Calibrations* (4 points).
Let $G \subset \mathbb{C}$ be open, bounded and non-empty subset of the complex plane with smooth boundary.

a) Show the following inequality for all $X, Y \in \mathbb{R}^{2n}$:

$$\omega_{\mathrm{std}}(X,Y)^2 \leq \langle X, X\rangle \langle Y, Y\rangle - \langle X, Y\rangle^2 ,$$

where ω_{std} is the standard symplectic form on $\mathbb{R}^{2n}$. Moreover, the equality case in the inequality case holds iff X, Y are linear dependent.

b) Let $F\colon \overline{G} \to \mathbb{C}^n$ be a smooth map. Show the *Wirtinger inequality*, i.e.

$$\int_G F^*\omega_{\mathrm{std}} \leq \mathrm{area}(F(G)) := \int_G \sqrt{\langle \partial_x F, \partial_x F\rangle \langle \partial_y F, \partial_y F\rangle - \langle \partial_x F, \partial_y F\rangle^2} \, \mathrm{d}x \, \mathrm{d}y.$$

The equality case holds iff the map F is holomorphic on G.

c) Let $H : \overline{G} \times [0,1] \to \mathbb{C}^n$ be a homotopy of smooth maps which fixes the boundary of G, i.e. $H(t,x) = F_0(x)$ for all $x \in \partial G$ and $t \in [0,1]$, where we write $F_i \coloneqq H(i,\cdot)$ for $i = 0,1$. Assume that F_0 is holomorphic on G. Show that:

$$\operatorname{area}(F_1(G)) \geq \operatorname{area}(F_0(G)).$$

Hint: One can use that the homotopy H given as above satisfies $\int_G F_0^*\omega_{std} = \int_G F_1^*\omega_{std}$.

Exercise 4: *Action-angle coordinates* (4 points).
Let $E = (E_1, \dots, E_k) \in \mathbb{R}^k$. We call a Hamiltonian system (M^{2k}, ω, H) *integrable* if there exist smooth function $H_1 = H, H_2, \dots, H_k$ on M, s.t.

i) The Poisson-brackets vanish, i.e. $\{H_i, H_j\} = 0$ for all $i, j \in \{1, \dots, k\}$,

ii) For all points $x \in N_E \coloneqq \{y \mid H_i(y) = E_i \text{ for all } i \in \{1, \dots, k\}\}$ the symplectic gradients $\operatorname{sgrad} H_1, \dots, \operatorname{sgrad} H_k$ are linear independent at x.

Assume that N_E is non-empty, compact and connected and let $x_0 \in N_E$.

a) Show that the map

$$F_{x_0}\colon \mathbb{R}^k \to M, (t_1, \dots, t_k) \mapsto \Phi_{t_1}^{H_1} \circ \dots \circ \Phi_{t_k}^{H_k}(x_0)$$

is well-defined and an immersion. Moreover the image of F is given by $\operatorname{image}(F) = N_E$.

b) The preimage $F_{x_0}^{-1}(x_0)$ is a discrete and closed subgroup of $\mathbb{R}^k$, which is generated by linear independent elements $v_1 \dots, v_k \in \mathbb{R}^k$.

c) Construct a diffeomorphism $G\colon T^k \coloneqq \mathbb{R}^k/\mathbb{Z}^k \to N_E$, s.t. there exist $w_0, w_1 \in \mathbb{R}^k$ with $\Phi_t^H(x_0) = G([w_0 + t w_1])$.

d) Let $C \coloneqq \overline{\{\Phi_t^H(x_0) \mid t \in \mathbb{R}\}}$ be the trajectory of the gradient flow of H. Show that $G^{-1}(C) \subset T^k$ is a closed submanifold. Determine all possible dimensions of C for different Hamiltonian systems (M, ω, H).