Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
Prof. Dr. Bernd Ammann, Jonathan Glöckle, Julian Seipel
Please hand in the exercises until **Monday, June 12th** in the lecture

Exercise Sheet no. 7

Exercise 1 (4 points).
Let (V, ω) be a $2n$-dimensional symplectic vector space and $L \subset V$ be a Lagrangian subspace.

a) Let $v_1, \dots, v_n$ be a basis of L. Show that there exist $w_1, \dots, w_n \in V$, s.t. $(v_1, \dots, v_n, w_1, \dots, w_n)$ is a symplectic basis of V, i. e.,

$$\omega(v_i, v_j) = \omega(w_i, w_j) = 0$$
$$\omega(v_i, w_j) = \delta_{ij}$$

holds.

b) Show that for every Lagrangian subspace $L \subset V$, there exists a *Lagrangian complement*, i. e., $L' \subset V$ a Lagrangian subspace with $L \oplus L' = V$.

c) We call a map $J: V \to V$ a compatible complex structure for ω if $J^2 = -\mathrm{id}_V$ holds and $g := \omega(\cdot, J\cdot)$ is a scalar product on V. Show that if $L \subset V$ is a Lagrangian subspace, then $L' := J(L)$ is Lagrangian complement for L.

Exercise 2: *Hamiltonian action* (4 points).
Let (M, ω) be a symplectic manifold. Let $H_1, \dots, H_k: M \to \mathbb{R}$ be Hamiltonian functions on M with compact support. We assume that

$$\{H_i, H_j\} = 0$$

holds for all $i, j = 1, \dots, k$.

a) Show the induced flows of the Hamiltonians commute, i. e.,

$$\Phi_t^{H_i} \circ \Phi_t^{H_j} = \Phi_t^{H_j} \circ \Phi_t^{H_i}.$$

b) Show that the following map is well-defined

$$\mathbb{R}^k \to \mathrm{Ham}_c(M, \omega)$$
$$(t_1, \dots, t_k) \mapsto \Phi_{t_1}^{H_1} \circ \dots \circ \Phi_{t_k}^{H_k},$$

and show that it is a group homomorphism.

Exercise 3 (4 points).
We consider the two-dimensional sphere as a symplectic manifold (S^2, ω_{S^2}), where the symplectic form is given by

$$\omega_{S^2,p}(v,w) = \langle p, v \times w \rangle_{\mathbb{R}^3}$$

with $p \in S^2$ and $v, w \in T_pS^2 = p^\perp$.

a) For $H_i = x_i$ with $i = 1,2,3$ determine the induced flows $\Phi_t^{H_i}$ for all times $t \in \mathbb{R}$.

b) Show that any element $A \in \mathrm{SO}(3)$ acts as a Hamiltonian diffeomorphism on (S^2, ω_{S^2}).

c) Let $x_i : S^2 \to \mathbb{R}$ be the coordinate functions of the sphere for $i = 1,2,3$. Show that the Poisson bracket of these functions satisfies

$$\{x_i, x_j\} = \epsilon_{ijk} x_k,$$

where ϵ_{ijk} is the Levi-Civita symbol. *Hint: Work in spherical coordinates.*

Exercise 4: *Harmonic oscillator* (4 points).
We consider the complex projective space $\mathbb{C}P^n$ and the following maps

$$i : S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$$
$$\pi : S^{2n+1} \to \mathbb{C}P^n$$

with i the inclusion and π the quotient map.

a) Let $H : \mathbb{C}^{n+1} \to \mathbb{R}, z \mapsto \frac{1}{2}\langle z, z \rangle_{\mathbb{C}^{n+1}}$. Show that $\operatorname{sgrad} H_{|z} = -i \cdot z$ holds.

b) Determine the trajectories of the Hamiltonian system $(\mathbb{C}^{n+1}, \omega_{\mathrm{st}}, H)$.

c) Show that there exists a unique symplectic form on $\mathbb{C}P^n$ called the Fubini-Study form, s.t. $i^*\omega_{\mathrm{st}} = \pi^*\omega_{\mathrm{FS}}$ holds.