Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
Prof. Dr. Bernd Ammann, Jonathan Glöckle, Julian Seipel
Please hand in the exercises until **Monday, June 5th** in the lecture

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Exercise Sheet no. 6

Exercise 1: *Electromagnetic field* (4 points).
The Lagrangian of a charged particle in an electromagnetic field is given by

$$L(x, v, t) := \frac{1}{2} m \|v\|^2 + e\langle A(x,t), v\rangle - e\phi(x,t),$$

where $m, e \in \mathbb{R}_{>0}$, $x, v \in \mathbb{R}^3$, $t \in (a,b)$, and both $A: \mathbb{R}^3 \times (a,b) \to \mathbb{R}^3$ and $\phi: \mathbb{R}^3 \times (a,b) \to \mathbb{R}$ are smooth.

a) Determine the Euler-Lagrange equation associated to L.

b) Calculate the Hamilton function belonging to L.

Exercise 2: *Symplectic maps* (4 points).
Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. Denote by $\pi_i: M_1 \times M_2 \to M_i$, $i = 1, 2$, the canonical projections. Let furthermore $f: M_1 \to M_2$ be a smooth map.

a) Show that $\omega_W := \pi_1^* \omega_1 - \pi_2^* \omega_2$ is a symplectic form on $W := M_1 \times M_2$.

b) We consider the graph of f,

$$\mathrm{Graph}(f) := \{(x, y) \in W \mid y = f(x)\} \subset W.$$

Show that the tangent space of $\mathrm{Graph}(f)$ in $(x, y) \in \mathrm{Graph}(f)$ is given by

$$T_{(x,y)}\mathrm{Graph}(f) = \{(v, d_x f(v)) \mid v \in T_x M_1\} \subset T_x M_1 \times T_y M_2 = T_{(x,y)} W.$$

Hint: Here, you may use without proof that for a smooth map $f: M \to N$ between smooth manifolds the following holds: The graph of f is a smooth submanifold of $M \times N$ and the map $\mathrm{id} \times f: M \to M \times N$, $x \mapsto (x, f(x))$ is a diffeomorphism onto the graph of f.

c) Conclude that the map $f : M_1 \to M_2$ is symplectic, i. e. $f^* \omega_2 = \omega_1$, if and only if $\mathrm{Graph}(f)$ is an isotropic submanifold of W, i. e. $T_{(x,y)}\mathrm{Graph}(f)$ is an isotropic subspace of $T_{(x,y)} W$ for all $(x, y) \in \mathrm{Graph}(f)$.

Exercise 3: *Poisson bracket* (4 points).
Let (M, ω) be a symplectic manifold. For two functions $f, g \in C^\infty(M)$, we define their *Poisson bracket* by $\{f, g\} := \omega(\mathrm{sgrad}\, f, \mathrm{sgrad}\, g) \in C^\infty(M)$, where sgrad denotes the symplectic gradient defined in Exercise 1 on Sheet 4.

a) Show that for any 2-form $\alpha \in \Omega^2(M)$ and all vector fields $X, Y, Z \in \Gamma(TM)$ the following formula holds:

$$\begin{aligned} \mathrm{d}\alpha(X, Y, Z) = \partial_X \alpha(Y, Z) + \partial_Y \alpha(Z, X) + \partial_Z \alpha(X, Y) \\ - \alpha([Y, Z], X) - \alpha([Z, X], Y) - \alpha([X, Y], Z). \end{aligned}$$

Hint: Apply Cartan's formula twice – once for 2-forms and once for 1-forms.

b) Prove that for all $f, g, h \in C^\infty(M)$

$$0 = -\mathrm{d}\omega(\mathrm{sgrad}\, f, \mathrm{sgrad}\, g, \mathrm{sgrad}\, h) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}.$$