Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
Prof. Dr. Bernd Ammann, Jonathan Glöckle, Julian Seipel
Please hand in the exercises until **Monday, May 22nd** in the lecture

Exercise Sheet no. 5

Exercise 1: *Symplectic orthogonal complement* (4 points).
Let (V,ω) be a $2n$-dimensional symplectic vector space, i. e. a $2n$-dimensional real vector space V together with an anti-symmetric non-degenerate bilinear form $\omega: V \times V \to \mathbb{R}$. Let $E \subset V$ be a linear subspace. We define the *symplectic orthogonal complement* of E in V as

$$E^{\perp_\omega} := \{v \in V \mid \omega(v,w) = 0 \text{ for all } w \in E\}.$$

Show the following:

a) $E^{\perp_\omega}$ is a linear subspace of V.

b) The following dimension formula holds: $\dim E + \dim E^{\perp_\omega} = 2n$.

c) $(E^{\perp_\omega})^{\perp_\omega} = E$.

Exercise 2: *Isotorpic, Lagrangian and symplectic subspaces* (4 points).
Let again (V,ω) be a $2n$-dimensional symplectic vector space. A linear subspace $E \subset V$ is called *isotropic* if $E \subset E^{\perp_\omega}$ and *Lagrangian* if $E = E^{\perp_\omega}$. It is *symplectic* if $E \cap E^{\perp_\omega} = \{0\}$. Show that the following holds fo any linear subspace $E \subset V$:

a) E is isotropic if and only if $\omega|_{E\times E} \equiv 0$. In particular, E is Lagrangian if and only if $\dim E = n$ and $\omega|_{E\times E} \equiv 0$.

b) E is symplectic if and only if $E^{\perp_\omega}$ is symplectic.

c) E is symplectic if and only if $E + E^{\perp_\omega} = V$.

d) E is symplectic if and only if the bilinear form $\omega|_{E\times E}$ is non-degenerate.

Exercise 3: *Legendre transformation geometrically* (4 points).
Let V be a finite-dimensional real vector space, $\Omega \subset V$ a convex open subset and $L: \Omega \to \mathbb{R}$ a smooth convex function. Assume that $\mathrm{d}L: \Omega \to \Omega^*, v \mapsto \mathrm{d}_v L$ is a diffeomorphism onto its image $\Omega^* \subset V^*$, so that its Legendre transformation $H = \mathbb{L}(L): \Omega^* \to \mathbb{R}$ is well-defined.

a) Show that for all $p \in \Omega^*$

$$H(p) = -\sup\Big\{c \in \mathbb{R} \mid p(v) + c \le L(v) \text{ for all } v \in \Omega\Big\}. \tag{1}$$

b) Graphically illustrate the procedure (1) for obtaining the Legendre transformation, in the case $\dim V = 1$.

c) Show that $\mathrm{Hess}_p H = (\mathrm{Hess}_v L)^{-1}$ for $p = \mathrm{d}_v L \in \Omega^*$.

Exercise 4: *Conserved quantities arising from Noether's theorem* (4 points).
For $n \in \mathbb{N}$, a function $E_{\text{pot}}: \mathbb{R}^n \to \mathbb{R}$ and a non-degenerate symmetric matrix $M \in \mathbb{R}^{n\times n}$, we consider the Lagrangian

$$\begin{aligned} L: T\mathbb{R}^n &\longrightarrow \mathbb{R} \\ T_q\mathbb{R}^n \ni (q,v) &\longmapsto \frac{1}{2}\langle v, Mv\rangle - E_{\text{pot}}(q). \end{aligned}$$

a) Assume that $n = 3k$, $M = \text{diag}(m_1 I_3, \ldots, m_k I_3)$ and E_{pot} is translationally symmetric in the following sense:

$$E_{pot}(q_1, \ldots q_k) = E_{pot}(q_1 + a, \ldots, q_k + a)$$

for all $q_1, \ldots q_k \in \mathbb{R}^3$ and all $a \in \mathbb{R}^3$. Determine the conserved momenta associated to the translational symmetry.

b) Assume that $n = 3$, $M = mI_3$ and E_{pot} is rotationally symmetric:

$$E_{pot}(q) = E_{pot}(Aq)$$

for all $q \in \mathbb{R}^3$ and all $A \in \text{SO}(3)$. Determine the conserved momenta associated to the rotational symmetry. Compare your result to Exercise 1 on Sheet 1.