Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
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Please hand in the exercises until **Monday, May 15th** in the lecture

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Exercise Sheet no. 4

Exercise 1: *Symplectic gradient* (4 points).
Let (M, ω) be a symplectic manifold, i. e. a pair of a smooth manifold M and a closed non-degenerate 2-form $\omega \in \Omega^2(M)$. Let furthermore $f \in C^\infty(M)$ be a smooth function on M.

a) Show that there exists a unique vector field $X \in \Gamma(TM)$, called *sympectic gradient* of f, such that $\mathrm{d}f = \omega(X, -)$.

b) Let $\phi: M \times \mathbb{R} \supset \mathcal{D} \to M$, $(p,t) \mapsto \phi_t(p)$ be a local flow of X (cf. Exercise 1 on Sheet 2). Show that ω is preserved by the flow, i. e. $(\phi_t^* \omega)_{|p} = \omega_{|p}$ for $(p,t) \in \mathcal{D}$.

Exercise 2: *Canonical symplectic structure* (4 points).
Let M be a smooth manifold and $\pi_{T^*M}: T^*M \to M$ be the cotangent bundle of M with its tangent bundle $\pi_{TT^*M}: TT^*M \to T^*M$. For $\alpha \in T_p^*M$ let $\mathrm{d}_\alpha \pi_{T^*M} : T_\alpha T^*M \to T_pM$ be the differential of π_{T^*M} at α. We define $\lambda_{\mathrm{can}} \in \Omega^1(T^*M)$ as follows: for $X \in T_\alpha T^*M$, with $\alpha \in T^*M$, we set

$$\lambda_{\mathrm{can}}(X) := \alpha(\mathrm{d}_\alpha \pi_{T^*M}(X)).$$

Denote by $q: M \supset U \to V \subset \mathbb{R}^n$ be a chart of M.

a) Show that

$$\begin{aligned}(p,q): \pi_{T^*M}^{-1}(U) &\longrightarrow \mathbb{R}^n \times V \\ \alpha &\longmapsto \left(\alpha\left(\frac{\partial}{\partial q^1}\right), \ldots, \alpha\left(\frac{\partial}{\partial q^n}\right), q^1(\pi_{T^*M}(\alpha)), \ldots, q^n(\pi_{T^*M}(\alpha))\right)\end{aligned}$$

defines a chart of T^*M.

b) Show that λ_{can} is well-defined and prove that $\lambda_{\mathrm{can}} = \sum_{i=1}^n p_i \mathrm{d}q^i$ in the chart $(p,q) = (p_1, \ldots, p_n, q^1, \ldots, q^n)$ defined above.

c) Prove that $\omega_{can} := -\mathrm{d}\lambda_{\mathrm{can}}$ is a symplectic form on T^*M and find its expression in the chart (p,q).

Exercise 3: *Another Legendre transformation* (4 points).
Let $(V, \langle -, - \rangle)$ be a finite dimensional Euclidean $\mathbb{R}$-vector space. Consider the function $L: V \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{k}\|x\|^k$ for $k > 1$. Show that $\mathrm{d}L: V \setminus \{0\} \to V^* \setminus \{0\}$, $x \mapsto \mathrm{d}_x L$ is a diffeomorphism and calculate $H: V^* \setminus \{0\} \to \mathbb{R}$, $p \mapsto \langle p, (\mathrm{d}L)^{-1}(p)\rangle - L((\mathrm{d}L)^{-1}(p))$.

Exercise 4: *Pullback connection* (4 points).
Let $V \to M$ be a smooth vector bundle of rank r with connection ∇ and let $F\colon N \to M$ be a smooth map. For a point $p \in N$, we choose a chart x of M and a local frame $(S_1, \dots, S_r)$ of π, both defined on an open neighborhood U around $F(p)$. For a section s of the pullback bundle $F^*V \to N$ and a vector $Y \in T_pN$, we now define

$$\nabla^F_Y s := \sum_{j=1}^{r} \left(\partial_Y s^j + \sum_{i=1}^{n} \Gamma^j_{ik}(F(p)) X^i s^k(p) \right) S_j(F(p))$$

where $s_{|F^{-1}(U)} = \sum_{j=1}^r s^j (S_j \circ F)$, $\mathrm{d}_pF(Y) = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}_{|F(p)}$ and $\nabla_{\frac{\partial}{\partial x^i}} S_k = \sum_{j=1}^r \Gamma^j_{ik} S_j$.

a) Show that ∇^F yields a well-defined connection on $F^*V \to N$.

b) Prove that $\nabla^F_Y (s \circ F) = (\nabla_X s) \circ F$ for all $s \in \Gamma(V)$, $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$ with $\mathrm{d}F(Y) = X \circ F$.

c) In the case $V = TM$, show that

$$\nabla^F_{\frac{\partial}{\partial y^i}} \mathrm{d}F\left(\frac{\partial}{\partial y^j}\right) - \nabla^F_{\frac{\partial}{\partial y^j}} \mathrm{d}F\left(\frac{\partial}{\partial y^i}\right) = T\left(\mathrm{d}F\left(\frac{\partial}{\partial y^i}\right), \mathrm{d}F\left(\frac{\partial}{\partial y^j}\right)\right),$$

where y is a chart of N and T is the torsion tensor (Exercise 2 on Sheet 3) of the connection ∇ on $TM \to M$.