Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
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Please hand in the exercises until **Monday, May 8th**

Exercise Sheet no. 3

Exercise 1: *Locality of connections* (4 points).
Let ∇ be a connection on a vector bundle $E \to M$ over a smooth manifold M and $U \subset M$ be an open subset. Show that $(\nabla_X s)_{|U} = (\nabla_{X'} s')_{|U}$ for all $X, X' \in \Gamma(TM)$ and $s, s' \in \Gamma(E)$ with $X_{|U} = X'_{|U}$ and $s_{|U} = s'_{|U}$.
Hint: You may take for granted that for all $p \in U$ there is a smooth function η with $\eta \equiv 1$ on a neighborhood of p and $\operatorname{supp}(\eta) \subset U$.

Exercise 2: *Torsion tensor* (4 points).
Let ∇ be a connection on $TM \to M$ for a smooth manifold M. We define its *torsion* $\mathcal{T}\colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ as $\mathcal{T}(X, Y) \coloneqq \nabla_X Y - \nabla_Y X - [X, Y]$.

a) Show that $\mathcal{T}(X, Y) = -\mathcal{T}(Y, X)$ and $\mathcal{T}(X + fX', Y) = \mathcal{T}(X, Y) + f\mathcal{T}(X', Y)$ for all $X, X', Y \in \Gamma(TM)$ and $f \in C^\infty(M)$.

b) Show that a tensor $T \in \Gamma(T^*M \otimes T^*M \otimes TM)$ exists with $\mathcal{T}(X, Y) = T(X, Y)$ for all $X, Y \in \Gamma(TM)$.

Exercise 3: *A Lagrangian* (4 points).
Consider the Lagrange function

$$L(v) = \frac{1}{4}\|v\|^4 - \frac{1}{2}\|v\|^2$$

on $\mathbb{R}^n$. For $x_1, x_2 \in \mathbb{R}^n$, determine all stationary points $q \in \mathcal{D}_{x_1,x_2} = \{q\colon [t_1, t_2] \to \mathbb{R}^n \text{ smooth} \mid q(t_1) = x_1, q(t_2) = x_2\}$ of the associated action functional $\mathcal{S}\colon \mathcal{D}_{x_1,x_2} \to \mathbb{R}$, $\mathcal{S}(q) = \int_{t_1}^{t_2} L(\dot{q})\mathrm{d}t$.
Hint: Show first that $\|\dot{q}\|^2$ is constant for the curves $q \in \mathcal{D}_{x_1,x_2}$ that are stationary for $\mathcal{S}$.

Exercise 4: *Legendre transformation* (4 points).
Let $(V, \|-\|)$ be a finite dimensional normed real vector space. Assume that $L\colon V \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{2}\|x\|^2$ is smooth and $\operatorname{Hess}_x L$ is positive definite for all $x \in V \setminus \{0\}$. Recall the definition of the dual norm on V^*: For $\alpha \in V^*$ it is given by $\|\alpha\|_* \coloneqq \sup_{x \in V \setminus \{0\}} \frac{\langle \alpha, x\rangle}{\|x\|}$. Here, and in the following, $\langle \alpha, x\rangle \coloneqq \alpha(x)$ denotes the duality pairing.

a) Show that $\mathrm{d}L\colon V \setminus \{0\} \to V^* \setminus \{0\}$, $x \to \mathrm{d}_x L$ is a well-defined local diffeomorphism.

b) Show that $\langle \mathrm{d}_x L, x\rangle = \|x\|^2 = \|\mathrm{d}_x L\|_*^2$ for all $x \in V \setminus \{0\}$.

c) Prove that $\mathrm{d}L$ is injective.
Hint: If $\mathrm{d}_x L = \mathrm{d}_y L$, consider $\langle \mathrm{d}_x L, tx + (1-t)y\rangle$ for $t \in [0, 1]$.

d) Conclude that $\mathrm{d}L$ is a diffeomorphism.

e) Consider

$$\begin{aligned} H\colon V^* \setminus \{0\} &\longrightarrow \mathbb{R} \\ p &\longmapsto \langle p, (\mathrm{d}L)^{-1}(p)\rangle - L((\mathrm{d}L)^{-1}(p)). \end{aligned}$$

Prove that $H(p) = \frac{1}{2}\|p\|_*^2$.