Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
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Please hand in the exercises until
Tuesday, May 2nd 1 pm, to **Mrs. Bonn, M 217**

Exercise Sheet no. 2

Exercise 1: *Vector flows* (4 points).
Let M be a manifold and X a smooth vector field on M. For $p \in M$, a *local flow* of X around p is a smooth map $\Phi^X : U \times (-\epsilon, \epsilon) \to M$, $(x,t) \mapsto \Phi^X_t(x)$, where $\epsilon \in (0, \infty]$ and U is an open neighborhood of p, with the following properties:

- $\Phi^X_0 = \mathrm{id}_M$
- $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^X_t = X \circ \Phi^X_t$.

a) Show that a local flow exists around any $p \in M$ and is unique up to restriction. Show, moreover, that Φ^X_t is a diffeomorphism onto its image for all $t \in (-\epsilon, \epsilon)$.
Hint: Reduce to an ODE on an open subset of $\mathbb{R}^n$.

b) Give an example of a vector field on $\mathbb{R}^n$, where the local flow around 0 does not extend to all times, i. e. where $\epsilon = \infty$ is not possible.

c) Assume now that M is closed. Prove that there exits a *global flow*, i. e. a local flow with $U = M$ and $\epsilon = \infty$.

Exercise 2: *Lie derivative* (4 points).
Recall the definition of pull-back of functions, differential 1-forms and vector fields: Along a smooth map $\phi : U \to V$ the pull-back of $f \in C^\infty(V)$ is $\phi^* f = f \circ \phi \in C^\infty(U)$ and the pull-back of $\alpha \in \Omega^1(V) = \Gamma(T^*V)$ is $\phi^*\alpha = \phi \circ \mathrm{d}\phi \in \Omega^1(U)$. If $\phi : U \to V$ is a local diffeomorphism, then the pull-back of a vector field $X \in \Gamma(TV)$ along ϕ is the unique vector field $\phi^* X \in \Gamma(TU)$ with $\mathrm{d}\phi \circ \phi^* X = X \circ \phi$. [1]

Let X be a smooth vector field on a manifold M. For a function, 1-form or vector field (or more generally some tensor field) T, the *Lie derivative* $\mathcal{L}_X T$ is defined as follows:

$$(\mathcal{L}_X T)_{|p} = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0}((\Phi^X_t)^* T)_{|p}$$

for all $p \in M$, where Φ^X is any local flow of X around p.

a) Show that $\mathcal{L}_X T$ is well-defined, i. e. smooth and independent of the choices made.

b) Prove that $\mathcal{L}_X f = \partial_X f$ for a function $f \in C^\infty(M)$.

c) Prove the product rule $\mathcal{L}_X(\alpha(Y)) = (\mathcal{L}_X \alpha)(Y) + \alpha(\mathcal{L}_X Y)$ for $\alpha \in \Omega^1(M)$ and $Y \in \Gamma(TM)$.

d) Show that $\mathcal{L}_X Y = [X, Y]$ for $Y \in \Gamma(TM)$, where the Lie bracket is defined through $\partial_{[X,Y]} f = (\partial_X \partial_Y - \partial_Y \partial_X) f$ for all $f \in C^\infty(M)$.

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[1] In a similar fashion, pull-backs of differential k-forms, for arbitrary k, and of (r, s)-tensor fields on M can be defined.

Exercise 3: *2-out-of-3 for the unitary group* (4 points).
Consider $\mathbb{C}^n$ with the standard scalar product $\langle -,-\rangle$, which can be decomposed as in Exercise 3 on Sheet 1 into real and imaginary part: $\langle -,-\rangle = g(-,-) + i\omega(-,-)$. Note that g and ω are the canonical real scalar product and the canonical symplectic form, respectively, on $\mathbb{C}^n$, which we canonically identify with $\mathbb{R}^{2n}$.

We consider the following groups:

$$\begin{aligned}
\mathrm{Gl}(2n,\mathbb{R}) &= \{T \in \mathbb{R}^{2n\times 2n} \mid \det(T) \neq 0\}\\
\mathrm{Gl}(n,\mathbb{C}) &= \{T \in \mathbb{C}^{n\times n} \mid \det(T) \neq 0\}\\
O(2n) &= \{T \in \mathbb{R}^{2n\times 2n} \mid g(T\cdot -, T\cdot -) = g\}\\
\mathrm{Symp}(2n) &= \{T \in \mathbb{R}^{2n\times 2n} \mid \omega(T\cdot -, T\cdot -) = \omega\}\\
U(n) &= \{T \in \mathbb{C}^{n\times n} \mid \langle T\cdot -, T\cdot -\rangle = \langle -,-\rangle\}.
\end{aligned}$$

Explain how all these groups (especially $\mathrm{Gl}(n,\mathbb{C})$ and $U(n)$) can be viewed as subgroups of $\mathrm{Gl}(2n,\mathbb{R})$ and show the following:

$$U(n) = \mathrm{Gl}(n,\mathbb{C}) \cap O(2n) = O(2n) \cap \mathrm{Symp}(2n) = \mathrm{Symp}(2n) \cap \mathrm{Gl}(n,\mathbb{C}).$$

Exercise 4: *Fundamental group of $U(n)$* (4 points).
Show that $\det\colon U(n) \to S^1$ induces an isomorphism on fundamental groups. You may use without proof that $SU(2n) = \{T \in U(n) \mid \det(T) = 1\}$ is simply connected.
Hint: You could start as follows. Let $\iota\colon S^1 \to U(n)$ be the map $x \to \begin{pmatrix} x & 0 \\ 0 & 1_{n-1} \end{pmatrix}$. For a loop $\gamma\colon S^1 \to U(n)$ now consider $x \mapsto (\iota \circ \det \circ \gamma(x))^{-1} \cdot \gamma(x)$ and show that this is null-homotopic. Alternative approaches might be even more efficient, depending on your previous knowledge.