Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
Prof. Dr. Bernd Ammann, Jonathan Glöckle, Julian Seipel
No submission – these exercises will be solved and discussed on **Wednesday, April 19th** during the exercise class

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Exercise Sheet no. 0

Exercise 1: *Differential forms* (0 points).
On $\mathbb{R}^4$, we consider the differential forms $\alpha \in \Omega^1\mathbb{R}^4 = \Gamma(T^*\mathbb{R}^4)$ and $\beta \in \Omega^2\mathbb{R}^4 = \Gamma(\bigwedge^2 T^*\mathbb{R}^4)$ given by

$$\alpha = dx^1 + x^2 dx^2,$$
$$\beta = \sin x^2 dx^1 \wedge dx^3 + \cos x^3 dx^2 \wedge dx^4.$$

(As usual, the dx^i are formed with respect to the chart $x = (x^1, \dots, x^4) = \mathrm{id}_{\mathbb{R}^4}$.) Calculate $\alpha \wedge \beta$ and $d\beta$.

Exercise 2: *Differentiation and insertion* (0 points).
Let M be a smooth manifold and $X \in \Gamma(TM)$ a vector field on M. Consider the graded ring of differential forms $\Omega^\bullet(M) = \Gamma(\bigwedge^\bullet T^*M)$ with its operations Cartan differentiation $\mathrm{d}\colon \Omega^\bullet(M) \to \Omega^\bullet(M)$ and X-insertion $\iota_X\colon \Omega^\bullet(M) \to \Omega^\bullet(M)$. Show that

$$\iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X = (\iota_X + \mathrm{d})^2$$

and that this operator is derivative (i. e. compatible with $\wedge$-product).

Exercise 3: *Cartan's magic formula* (0 points).
Let again M be a smooth manifold and $X \in \Gamma(TM)$ be a vector field on M. If Φ_X is the local flow of X, then the *Lie-derivative* on differential forms is defined by

$$\mathcal{L}_X\colon \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M),$$
$$\omega \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \Phi_X(t)^*\omega.$$

Show that $\mathcal{L}_X = \iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X$.