Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023

Prof. Dr. Bernd Ammann, Jonathan Glöckle, Julian Seipel

No submission – these exercises will be solved and discussed on **Wednesday, April 19th** during the exercise class

Exercise Sheet no. 0

Exercise 1: *Differential forms* (0 points).
On $\mathbb{R}^4$, we consider the differential forms $\alpha \in \Omega^1\mathbb{R}^4 = \Gamma(T^*\mathbb{R}^4)$ and $\beta \in \Omega^2\mathbb{R}^4 = \Gamma(\bigwedge^2 T^*\mathbb{R}^4)$ given by

$$\alpha = dx^1 + x^2 dx^2,$$
$$\beta = \sin x^2 dx^1 \wedge dx^3 + \cos x^3 dx^2 \wedge dx^4.$$

(As usual, the dx^i are formed with respect to the chart $x = (x^1, \dots, x^4) = \mathrm{id}_{\mathbb{R}^4}$.) Calculate $\alpha \wedge \beta$ and $d\beta$.

Exercise 2: *Differentiation and insertion* (0 points).
Let M be a smooth manifold and $X \in \Gamma(TM)$ a vector field on M. Consider the graded ring of differential forms $\Omega^\bullet(M) = \Gamma(\bigwedge^\bullet T^*M)$ with its operations Cartan differentiation $\mathrm{d}\colon \Omega^\bullet(M) \to \Omega^\bullet(M)$ and X-insertion $\iota_X\colon \Omega^\bullet(M) \to \Omega^\bullet(M)$. Show that

$$\iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X = (\iota_X + \mathrm{d})^2$$

and that this operator is derivative (i. e. compatible with $\wedge$-product).

Exercise 3: *Cartan's magic formula* (0 points).
Let again M be a smooth manifold and $X \in \Gamma(TM)$ be a vector field on M. If Φ_X is the local flow of X, then the *Lie-derivative* on differential forms is defined by

$$\mathcal{L}_X\colon \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M),$$
$$\omega \longmapsto \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \Phi_X(t)^*\omega.$$

Show that $\mathcal{L}_X = \iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X$.

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Please hand in the exercises until **Monday, April 24th**

Exercise Sheet no. 1

Exercise 1: *Angular momentum* (4 points).
We consider the Newtonian equation of motion

$$m\ddot{\mathbf{x}}(t) = -\mathrm{grad} V(\mathbf{x}(t)),$$

where $m \in \mathbb{R}$, $\mathbf{x}\colon (a,b) \to \mathbb{R}^3$ and $V(x) = f(\|x\|)$ for a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ and $\|\cdot\|$ the standard norm on $\mathbb{R}^3$. We define the angular momentum as $\mathbf{L}(t) \coloneqq m\mathbf{x}(t) \times \dot{\mathbf{x}}(t)$. Show that angular momentum is conserved, i. e. $\frac{d}{dt}\mathbf{L}(t) = 0$ for all $t \in (a,b)$ and any solution $\mathbf{x}(t)$ of the equation of motion.

Exercise 2: *Symplectic basis* (4 points).
Let V be a finite-dimensional real vector space. Furthermore, let $\omega\colon V \times V \to \mathbb{R}$ be an anti-symmetric non-degenerate bilinear form.[1] Show: There is an $n \in \mathbb{N}$ and a basis $(u_1, \dots, u_n, v_1, \dots, v_n)$ of V such that

$$\omega(u_j, u_k) = \omega(v_j, v_k) = 0, \qquad \omega(u_j, v_k) = \delta_{jk}$$

for all $j, k \in \{1, \dots, n\}$.
Hint: First construct suitable u_1 and v_1, and then consider

$$V_1 \coloneqq \{x \in V \mid \omega(x, u_1) = \omega(x, v_1) = 0\}.$$

Remark: In particular, this exercise shows that the dimension of V is even, $\dim V = 2n$.

Exercise 3: *Hermitian scalar products* (4 points).
Let V be complex vector space and g a real scalar product on (the underlying real vector space of) V. Define $\omega(X,Y) \coloneqq g(iX, Y)$ for all $X, Y \in V$.

a) Show that ω is a non-degenerate bilinear form.

b) Prove that the following are equivalent:

   i) $g(iX, iY) = g(X,Y)$ for all $X, Y \in V$.
   ii) $\omega(X,Y) = -\omega(Y,X)$ for all $X, Y \in V$.
   iii) $g + i\omega$ is a Hermitian scalar product on V. [2]

Exercise 4: *A special linear group* (4 points).
Let $\mathrm{SL}(2,\mathbb{R})$ be the set of all 2×2-matrices with determinant 1. Show that $\mathrm{SL}(2,\mathbb{R})$ is a submanifold of $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$. Prove that $\mathrm{SL}(2,\mathbb{R})$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

[1] Recall: A bilinear form is called *anti-symmetric*, if $\omega(x,y) = -\omega(y,x)$ for all $x, y \in V$. A bilinear form is called *non-degenerate*, if $\omega(x,y) = 0$ for all $x \in V$ implies $y = 0$.

[2] We use the convention that Hermitian scalar products are $\mathbb{C}$-linear in the second argument.

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Please hand in the exercises until
Tuesday, May 2nd 1 pm, to **Mrs. Bonn, M 217**

Exercise Sheet no. 2

Exercise 1: *Vector flows* (4 points).
Let M be a manifold and X a smooth vector field on M. For $p \in M$, a *local flow* of X around p is a smooth map $\Phi^X : U \times (-\epsilon, \epsilon) \to M$, $(x,t) \mapsto \Phi^X_t(x)$, where $\epsilon \in (0, \infty]$ and U is an open neighborhood of p, with the following properties:

- $\Phi^X_0 = \mathrm{id}_M$
- $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^X_t = X \circ \Phi^X_t$.

a) Show that a local flow exists around any $p \in M$ and is unique up to restriction. Show, moreover, that Φ^X_t is a diffeomorphism onto its image for all $t \in (-\epsilon, \epsilon)$.
Hint: Reduce to an ODE on an open subset of $\mathbb{R}^n$.

b) Give an example of a vector field on $\mathbb{R}^n$, where the local flow around 0 does not extend to all times, i. e. where $\epsilon = \infty$ is not possible.

c) Assume now that M is closed. Prove that there exits a *global flow*, i. e. a local flow with $U = M$ and $\epsilon = \infty$.

Exercise 2: *Lie derivative* (4 points).
Recall the definition of pull-back of functions, differential 1-forms and vector fields: Along a smooth map $\phi : U \to V$ the pull-back of $f \in C^\infty(V)$ is $\phi^* f = f \circ \phi \in C^\infty(U)$ and the pull-back of $\alpha \in \Omega^1(V) = \Gamma(T^*V)$ is $\phi^*\alpha = \phi \circ \mathrm{d}\phi \in \Omega^1(U)$. If $\phi : U \to V$ is a local diffeomorphism, then the pull-back of a vector field $X \in \Gamma(TV)$ along ϕ is the unique vector field $\phi^* X \in \Gamma(TU)$ with $\mathrm{d}\phi \circ \phi^* X = X \circ \phi$. [1]

Let X be a smooth vector field on a manifold M. For a function, 1-form or vector field (or more generally some tensor field) T, the *Lie derivative* $\mathcal{L}_X T$ is defined as follows:

$$(\mathcal{L}_X T)_{|p} = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} ((\Phi^X_t)^* T)_{|p}$$

for all $p \in M$, where Φ^X is any local flow of X around p.

a) Show that $\mathcal{L}_X T$ is well-defined, i. e. smooth and independent of the choices made.

b) Prove that $\mathcal{L}_X f = \partial_X f$ for a function $f \in C^\infty(M)$.

c) Prove the product rule $\mathcal{L}_X(\alpha(Y)) = (\mathcal{L}_X \alpha)(Y) + \alpha(\mathcal{L}_X Y)$ for $\alpha \in \Omega^1(M)$ and $Y \in \Gamma(TM)$.

d) Show that $\mathcal{L}_X Y = [X, Y]$ for $Y \in \Gamma(TM)$, where the Lie bracket is defined through $\partial_{[X,Y]} f = (\partial_X \partial_Y - \partial_Y \partial_X) f$ for all $f \in C^\infty(M)$.

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[1] In a similar fashion, pull-backs of differential k-forms, for arbitrary k, and of (r, s)-tensor fields on M can be defined.

Exercise 3: *2-out-of-3 for the unitary group* (4 points).
Consider $\mathbb{C}^n$ with the standard scalar product $\langle -,- \rangle$, which can be decomposed as in Exercise 3 on Sheet 1 into real and imaginary part: $\langle -,- \rangle = g(-,-) + i\omega(-,-)$. Note that g and ω are the canonical real scalar product and the canonical symplectic form, respectively, on $\mathbb{C}^n$, which we canonically identify with $\mathbb{R}^{2n}$.

We consider the following groups:

$$\begin{aligned} \mathrm{Gl}(2n,\mathbb{R}) &= \{T \in \mathbb{R}^{2n\times 2n} \mid \det(T) \neq 0\} \\ \mathrm{Gl}(n,\mathbb{C}) &= \{T \in \mathbb{C}^{n\times n} \mid \det(T) \neq 0\} \\ O(2n) &= \{T \in \mathbb{R}^{2n\times 2n} \mid g(T\cdot -, T\cdot -) = g\} \\ \mathrm{Symp}(2n) &= \{T \in \mathbb{R}^{2n\times 2n} \mid \omega(T\cdot -, T\cdot -) = \omega\} \\ U(n) &= \{T \in \mathbb{C}^{n\times n} \mid \langle T\cdot -, T\cdot - \rangle = \langle -,- \rangle\}. \end{aligned}$$

Explain how all these groups (especially $\mathrm{Gl}(n,\mathbb{C})$ and $U(n)$) can be viewed as subgroups of $\mathrm{Gl}(2n,\mathbb{R})$ and show the following:

$$U(n) = \mathrm{Gl}(n,\mathbb{C}) \cap O(2n) = O(2n) \cap \mathrm{Symp}(2n) = \mathrm{Symp}(2n) \cap \mathrm{Gl}(n,\mathbb{C}).$$

Exercise 4: *Fundamental group of $U(n)$* (4 points).
Show that $\det\colon U(n) \to S^1$ induces an isomorphism on fundamental groups. You may use without proof that $SU(2n) = \{T \in U(n) \mid \det(T) = 1\}$ is simply connected.
Hint: You could start as follows. Let $\iota\colon S^1 \to U(n)$ be the map $x \to \begin{pmatrix} x & 0 \\ 0 & 1_{n-1} \end{pmatrix}$. For a loop $\gamma\colon S^1 \to U(n)$ now consider $x \mapsto (\iota \circ \det \circ \gamma(x))^{-1} \cdot \gamma(x)$ and show that this is null-homotopic. Alternative approaches might be even more efficient, depending on your previous knowledge.

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Exercise Sheet no. 3

Exercise 1: *Locality of connections* (4 points).
Let ∇ be a connection on a vector bundle $E \to M$ over a smooth manifold M and $U \subset M$ be an open subset. Show that $(\nabla_X s)_{|U} = (\nabla_{X'} s')_{|U}$ for all $X, X' \in \Gamma(TM)$ and $s, s' \in \Gamma(E)$ with $X_{|U} = X'_{|U}$ and $s_{|U} = s'_{|U}$.
Hint: You may take for granted that for all $p \in U$ there is a smooth function η with $\eta \equiv 1$ on a neighborhood of p and $\operatorname{supp}(\eta) \subset U$.

Exercise 2: *Torsion tensor* (4 points).
Let ∇ be a connection on $TM \to M$ for a smooth manifold M. We define its *torsion* $\mathcal{T}: \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ as $\mathcal{T}(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$.

a) Show that $\mathcal{T}(X,Y) = -\mathcal{T}(Y,X)$ and $\mathcal{T}(X + fX', Y) = \mathcal{T}(X,Y) + f\mathcal{T}(X',Y)$ for all $X, X', Y \in \Gamma(TM)$ and $f \in C^\infty(M)$.

b) Show that a tensor $T \in \Gamma(T^*M \otimes T^*M \otimes TM)$ exists with $\mathcal{T}(X,Y) = T(X,Y)$ for all $X, Y \in \Gamma(TM)$.

Exercise 3: *A Lagrangian* (4 points).
Consider the Lagrange function

$$L(v) = \frac{1}{4}\|v\|^4 - \frac{1}{2}\|v\|^2$$

on $\mathbb{R}^n$. For $x_1, x_2 \in \mathbb{R}^n$, determine all stationary points $q \in \mathcal{D}_{x_1,x_2} = \{q: [t_1, t_2] \to \mathbb{R}^n \text{ smooth} \mid q(t_1) = x_1, q(t_2) = x_2\}$ of the associated action functional $\mathcal{S}: \mathcal{D}_{x_1,x_2} \to \mathbb{R}$, $\mathcal{S}(q) = \int_{t_1}^{t_2} L(\dot q)\mathrm{d}t$.
Hint: Show first that $\|\dot q\|^2$ is constant for the curves $q \in \mathcal{D}_{x_1,x_2}$ that are stationary for $\mathcal{S}$.

Exercise 4: *Legendre transformation* (4 points).
Let $(V, \|-\|)$ be a finite dimensional normed real vector space. Assume that $L: V \smallsetminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{2}\|x\|^2$ is smooth and $\operatorname{Hess}_x L$ is positive definite for all $x \in V \smallsetminus \{0\}$. Recall the definition of the dual norm on V^*: For $\alpha \in V^*$ it is given by $\|\alpha\|_* := \sup_{x \in V \smallsetminus \{0\}} \frac{\langle \alpha, x\rangle}{\|x\|}$. Here, and in the following, $\langle \alpha, x\rangle := \alpha(x)$ denotes the duality pairing.

a) Show that $\mathrm{d}L: V \smallsetminus \{0\} \to V^* \smallsetminus \{0\}$, $x \to \mathrm{d}_x L$ is a well-defined local diffeomorphism.

b) Show that $\langle \mathrm{d}_x L, x\rangle = \|x\|^2 = \|\mathrm{d}_x L\|_*^2$ for all $x \in V \smallsetminus \{0\}$.

c) Prove that $\mathrm{d}L$ is injective.
Hint: If $\mathrm{d}_x L = \mathrm{d}_y L$, consider $\langle \mathrm{d}_x L, tx + (1-t)y\rangle$ for $t \in [0,1]$.

d) Conclude that $\mathrm{d}L$ is a diffeomorphism.

e) Consider

$$\begin{aligned} H: V^* \smallsetminus \{0\} &\longrightarrow \mathbb{R} \\ p &\longmapsto \langle p, (\mathrm{d}L)^{-1}(p)\rangle - L((\mathrm{d}L)^{-1}(p)). \end{aligned}$$

Prove that $H(p) = \frac{1}{2}\|p\|_*^2$.

Exercise Sheet no. 4

Exercise 1: *Symplectic gradient* (4 points).
Let (M,ω) be a symplectic manifold, i. e. a pair of a smooth manifold M and a closed non-degenerate 2-form $\omega \in \Omega^2(M)$. Let furthermore $f \in C^\infty(M)$ be a smooth function on M.

a) Show that there exists a unique vector field $X \in \Gamma(TM)$, called *sympectic gradient* of f, such that $\mathrm{d}f = \omega(X,-)$.

b) Let $\phi\colon M \times \mathbb{R} \supset \mathcal{D} \to M$, $(p,t) \mapsto \phi_t(p)$ be a local flow of X (cf. Exercise 1 on Sheet 2). Show that ω is preserved by the flow, i. e. $(\phi_t^*\omega)_{|p} = \omega_{|p}$ for $(p,t) \in \mathcal{D}$.

Exercise 2: *Canonical symplectic structure* (4 points).
Let M be a smooth manifold and $\pi_{T^*M}\colon T^*M \to M$ be the cotangent bundle of M with its tangent bundle $\pi_{TT^*M}\colon TT^*M \to T^*M$. For $\alpha \in T_p^*M$ let $\mathrm{d}_\alpha\pi_{T^*M} : T_\alpha T^*M \to T_pM$ be the differential of π_{T^*M} at α. We define $\lambda_{\mathrm{can}} \in \Omega^1(T^*M)$ as follows: for $X \in T_\alpha T^*M$, with $\alpha \in T^*M$, we set

$$\lambda_{\mathrm{can}}(X) \coloneqq \alpha(\mathrm{d}_\alpha\pi_{T^*M}(X)).$$

Denote by $q\colon M \supset U \to V \subset \mathbb{R}^n$ be a chart of M.

a) Show that

$$\begin{aligned}(p,q)\colon \pi_{T^*M}^{-1}(U) &\longrightarrow \mathbb{R}^n \times V\\ \alpha &\longmapsto \left(\alpha\left(\frac{\partial}{\partial q^1}\right),\dots,\alpha\left(\frac{\partial}{\partial q^n}\right), q^1(\pi_{T^*M}(\alpha)),\dots,q^n(\pi_{T^*M}(\alpha))\right)\end{aligned}$$

defines a chart of T^*M.

b) Show that λ_{can} is well-defined and prove that $\lambda_{\mathrm{can}} = \sum_{i=1}^n p_i \mathrm{d}q^i$ in the chart $(p,q) = (p_1,\dots,p_n,q^1,\dots,q^n)$ defined above.

c) Prove that $\omega_{can} \coloneqq -\mathrm{d}\lambda_{\mathrm{can}}$ is a symplectic form on T^*M and find its expression in the chart (p,q).

Exercise 3: *Another Legendre transformation* (4 points).
Let $(V,\langle -,-\rangle)$ be a finite dimensional Euclidean $\mathbb{R}$-vector space. Consider the function $L\colon V \setminus \{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{k}\|x\|^k$ for $k > 1$. Show that $\mathrm{d}L\colon V \setminus \{0\} \to V^* \setminus \{0\}$, $x \mapsto \mathrm{d}_xL$ is a diffeomorphism and calculate $H\colon V^* \setminus \{0\} \to \mathbb{R}$, $p \mapsto \langle p, (\mathrm{d}L)^{-1}(p)\rangle - L((\mathrm{d}L)^{-1}(p))$.

Exercise 4: *Pullback connection* (4 points).
Let $V \to M$ be a smooth vector bundle of rank r with connection ∇ and let $F\colon N \to M$ be a smooth map. For a point $p \in N$, we choose a chart x of M and a local frame $(S_1, \dots, S_r)$ of π, both defined on an open neighborhood U around $F(p)$. For a section s of the pullback bundle $F^*V \to N$ and a vector $Y \in T_pN$, we now define

$$\nabla^F_Y s := \sum_{j=1}^{r} \left(\partial_Y s^j + \sum_{i=1}^{n} \Gamma^j_{ik}(F(p)) X^i s^k(p) \right) S_j(F(p))$$

where $s_{|F^{-1}(U)} = \sum_{j=1}^r s^j (S_j \circ F)$, $\mathrm{d}_pF(Y) = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}_{|F(p)}$ and $\nabla_{\frac{\partial}{\partial x^i}} S_k = \sum_{j=1}^r \Gamma^j_{ik} S_j$.

a) Show that ∇^F yields a well-defined connection on $F^*V \to N$.

b) Prove that $\nabla^F_Y(s \circ F) = (\nabla_X s) \circ F$ for all $s \in \Gamma(V)$, $X \in \Gamma(TM)$ and $Y \in \Gamma(TN)$ with $\mathrm{d}F(Y) = X \circ F$.

c) In the case $V = TM$, show that

$$\nabla^F_{\frac{\partial}{\partial y^i}} \mathrm{d}F\left(\frac{\partial}{\partial y^j}\right) - \nabla^F_{\frac{\partial}{\partial y^j}} \mathrm{d}F\left(\frac{\partial}{\partial y^i}\right) = T\left(\mathrm{d}F\left(\frac{\partial}{\partial y^i}\right), \mathrm{d}F\left(\frac{\partial}{\partial y^j}\right)\right),$$

where y is a chart of N and T is the torsion tensor (Exercise 2 on Sheet 3) of the connection ∇ on $TM \to M$.

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Please hand in the exercises until **Monday, May 22nd** in the lecture

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Exercise Sheet no. 5

Exercise 1: *Symplectic orthogonal complement* (4 points).
Let (V, ω) be a $2n$-dimensional symplectic vector space, i. e. a $2n$-dimensional real vector space V together with an anti-symmetric non-degenerate bilinear form $\omega: V \times V \to \mathbb{R}$. Let $E \subset V$ be a linear subspace. We define the *symplectic orthogonal complement* of E in V as

$$E^{\perp_\omega} := \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in E\}.$$

Show the following:

a) $E^{\perp_\omega}$ is a linear subspace of V.

b) The following dimension formula holds: $\dim E + \dim E^{\perp_\omega} = 2n$.

c) $(E^{\perp_\omega})^{\perp_\omega} = E$.

Exercise 2: *Isotorpic, Lagrangian and symplectic subspaces* (4 points).
Let again (V, ω) be a $2n$-dimensional symplectic vector space. A linear subspace $E \subset V$ is called *isotropic* if $E \subset E^{\perp_\omega}$ and *Lagrangian* if $E = E^{\perp_\omega}$. It is *symplectic* if $E \cap E^{\perp_\omega} = \{0\}$. Show that the following holds fo any linear subspace $E \subset V$:

a) E is isotropic if and only if $\omega|_{E\times E} \equiv 0$. In particular, E is Lagrangian if and only if $\dim E = n$ and $\omega|_{E\times E} \equiv 0$.

b) E is symplectic if and only if $E^{\perp_\omega}$ is symplectic.

c) E is symplectic if and only if $E + E^{\perp_\omega} = V$.

d) E is symplectic if and only if the bilinear form $\omega|_{E\times E}$ is non-degenerate.

Exercise 3: *Legendre transformation geometrically* (4 points).
Let V be a finite-dimensional real vector space, $\Omega \subset V$ a convex open subset and $L: \Omega \to \mathbb{R}$ a smooth convex function. Assume that $\mathrm{d}L: \Omega \to \Omega^*, v \mapsto \mathrm{d}_v L$ is a diffeomorphism onto its image $\Omega^* \subset V^*$, so that its Legendre transformation $H = \mathbb{L}(L): \Omega^* \to \mathbb{R}$ is well-defined.

a) Show that for all $p \in \Omega^*$

$$H(p) = -\sup\Big\{c \in \mathbb{R} \mid p(v) + c \le L(v) \text{ for all } v \in \Omega\Big\}. \tag{1}$$

b) Graphically illustrate the procedure (1) for obtaining the Legendre transformation, in the case $\dim V = 1$.

c) Show that $\mathrm{Hess}_p\, H = (\mathrm{Hess}_v\, L)^{-1}$ for $p = \mathrm{d}_v L \in \Omega^*$.

Exercise 4: *Conserved quantities arising from Noether's theorem* (4 points).
For $n \in \mathbb{N}$, a function $E_{\text{pot}}: \mathbb{R}^n \to \mathbb{R}$ and a non-degenerate symmetric matrix $M \in \mathbb{R}^{n \times n}$, we consider the Lagrangian

$$\begin{aligned} L: T\mathbb{R}^n &\longrightarrow \mathbb{R} \\ T_q\mathbb{R}^n \ni (q,v) &\longmapsto \frac{1}{2}\langle v, Mv\rangle - E_{\text{pot}}(q). \end{aligned}$$

a) Assume that $n = 3k$, $M = \operatorname{diag}(m_1 I_3, \ldots, m_k I_3)$ and E_{pot} is translationally symmetric in the following sense:

$$E_{pot}(q_1, \ldots q_k) = E_{pot}(q_1 + a, \ldots, q_k + a)$$

for all $q_1, \ldots q_k \in \mathbb{R}^3$ and all $a \in \mathbb{R}^3$. Determine the conserved momenta associated to the translational symmetry.

b) Assume that $n = 3$, $M = mI_3$ and E_{pot} is rotationally symmetric:

$$E_{pot}(q) = E_{pot}(Aq)$$

for all $q \in \mathbb{R}^3$ and all $A \in \mathrm{SO}(3)$. Determine the conserved momenta associated to the rotational symmetry. Compare your result to Exercise 1 on Sheet 1.

Exercise Sheet no. 6

Exercise 1: *Electromagnetic field* (4 points).
The Lagrangian of a charged particle in an electromagnetic field is given by

$$L(x, v, t) := \frac{1}{2} m \|v\|^2 + e\langle A(x, t), v\rangle - e\phi(x, t),$$

where $m, e \in \mathbb{R}_{>0}$, $x, v \in \mathbb{R}^3$, $t \in (a, b)$, and both $A\colon \mathbb{R}^3 \times (a, b) \to \mathbb{R}^3$ and $\phi\colon \mathbb{R}^3 \times (a, b) \to \mathbb{R}$ are smooth.

a) Determine the Euler-Lagrange equation associated to L.

b) Calculate the Hamilton function belonging to L.

Exercise 2: *Symplectic maps* (4 points).
Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. Denote by $\pi_i\colon M_1 \times M_2 \to M_i$, $i = 1, 2$, the canonical projections. Let furthermore $f\colon M_1 \to M_2$ be a smooth map.

a) Show that $\omega_W := \pi_1^*\omega_1 - \pi_2^*\omega_2$ is a symplectic form on $W := M_1 \times M_2$.

b) We consider the graph of f,

$$\mathrm{Graph}(f) := \{(x, y) \in W \mid y = f(x)\} \subset W.$$

Show that the tangent space of $\mathrm{Graph}(f)$ in $(x, y) \in \mathrm{Graph}(f)$ is given by

$$T_{(x,y)}\mathrm{Graph}(f) = \{(v, d_x f(v)) \mid v \in T_x M_1\} \subset T_x M_1 \times T_y M_2 = T_{(x,y)}W.$$

Hint: Here, you may use without proof that for a smooth map $f\colon M \to N$ between smooth manifolds the following holds: The graph of f is a smooth submanifold of $M \times N$ and the map $\mathrm{id} \times f\colon M \to M \times N$, $x \mapsto (x, f(x))$ is a diffeomorphism onto the graph of f.

c) Conclude that the map $f : M_1 \to M_2$ is symplectic, i. e. $f^*\omega_2 = \omega_1$, if and only if $\mathrm{Graph}(f)$ is an isotropic submanifold of W, i. e. $T_{(x,y)}\mathrm{Graph}(f)$ is an isotropic subspace of $T_{(x,y)}W$ for all $(x, y) \in \mathrm{Graph}(f)$.

Exercise 3: *Poisson bracket* (4 points).
Let (M, ω) be a symplectic manifold. For two functions $f, g \in C^\infty(M)$, we define their *Poisson bracket* by $\{f, g\} := \omega(\mathrm{sgrad}\, f, \mathrm{sgrad}\, g) \in C^\infty(M)$, where sgrad denotes the symplectic gradient defined in Exercise 1 on Sheet 4.

a) Show that for any 2-form $\alpha \in \Omega^2(M)$ and all vector fields $X, Y, Z \in \Gamma(TM)$ the following formula holds:

$$\begin{aligned} \mathrm{d}\alpha(X, Y, Z) = \partial_X \alpha(Y, Z) + \partial_Y \alpha(Z, X) + \partial_Z \alpha(X, Y) \\ - \alpha([Y, Z], X) - \alpha([Z, X], Y) - \alpha([X, Y], Z). \end{aligned}$$

Hint: Apply Cartan's formula twice – once for 2-forms and once for 1-forms.

b) Prove that for all $f, g, h \in C^\infty(M)$

$$0 = -\mathrm{d}\omega(\mathrm{sgrad}\, f, \mathrm{sgrad}\, g, \mathrm{sgrad}\, h) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}.$$

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Please hand in the exercises until **Monday, June 12th** in the lecture

Exercise Sheet no. 7

Exercise 1 (4 points).
Let (V, ω) be a $2n$-dimensional symplectic vector space and $L \subset V$ be a Lagrangian subspace.

a) Let $v_1, \ldots, v_n$ be a basis of L. Show that there exist $w_1, \ldots, w_n \in V$, s.t. $(v_1, \ldots, v_n, w_1, \ldots, w_n)$ is a symplectic basis of V, i. e.,

$$\omega(v_i, v_j) = \omega(w_i, w_j) = 0$$
$$\omega(v_i, w_j) = \delta_{ij}$$

holds.

b) Show that for every Lagrangian subspace $L \subset V$, there exists a *Lagrangian complement*, i. e., $L' \subset V$ a Lagrangian subspace with $L \oplus L' = V$.

c) We call a map $J: V \to V$ a compatible complex structure for ω if $J^2 = -\mathrm{id}_V$ holds and $g := \omega(\cdot, J\cdot)$ is a scalar product on V. Show that if $L \subset V$ is a Lagrangian subspace, then $L' := J(L)$ is Lagrangian complement for L.

Exercise 2: *Hamiltonian action* (4 points).
Let (M, ω) be a symplectic manifold. Let $H_1, \ldots, H_k: M \to \mathbb{R}$ be Hamiltonian functions on M with compact support. We assume that

$$\{H_i, H_j\} = 0$$

holds for all $i, j = 1, \ldots, k$.

a) Show the induced flows of the Hamiltonians commute, i. e.,

$$\Phi_t^{H_i} \circ \Phi_t^{H_j} = \Phi_t^{H_j} \circ \Phi_t^{H_i}.$$

b) Show that the following map is well-defined

$$\mathbb{R}^k \to \mathrm{Ham}_c(M, \omega)$$
$$(t_1, \ldots, t_k) \mapsto \Phi_{t_1}^{H_1} \circ \ldots \circ \Phi_{t_k}^{H_k},$$

and show that it is a group homomorphism.

Exercise 3 (4 points).
We consider the two-dimensional sphere as a symplectic manifold (S^2, ω_{S^2}), where the symplectic form is given by

$$\omega_{S^2,p}(v,w) = \langle p, v \times w \rangle_{\mathbb{R}^3}$$

with $p \in S^2$ and $v, w \in T_pS^2 = p^\perp$.

a) For $H_i = x_i$ with $i = 1,2,3$ determine the induced flows $\Phi_t^{H_i}$ for all times $t \in \mathbb{R}$.

b) Show that any element $A \in \mathrm{SO}(3)$ acts as a Hamiltonian diffeomorphism on (S^2, ω_{S^2}).

c) Let $x_i\colon S^2 \to \mathbb{R}$ be the coordinate functions of the sphere for $i = 1,2,3$. Show that the Poisson bracket of these functions satisfies

$$\{x_i, x_j\} = \epsilon_{ijk} x_k,$$

where ϵ_{ijk} is the Levi-Civita symbol. *Hint: Work in spherical coordinates.*

Exercise 4: *Harmonic oscillator* (4 points).
We consider the complex projective space $\mathbb{C}P^n$ and the following maps

$$i\colon S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$$
$$\pi\colon S^{2n+1} \to \mathbb{C}P^n$$

with i the inclusion and π the quotient map.

a) Let $H\colon \mathbb{C}^{n+1} \to \mathbb{R}, z \mapsto \frac{1}{2}\langle z, z\rangle_{\mathbb{C}^{n+1}}$. Show that $\operatorname{sgrad} H_{|z} = -i \cdot z$ holds.

b) Determine the trajectories of the Hamiltonian system $(\mathbb{C}^{n+1}, \omega_{\mathrm{st}}, H)$.

c) Show that there exists a unique symplectic form on $\mathbb{C}P^n$ called the Fubini-Study form, s.t. $i^*\omega_{\mathrm{st}} = \pi^*\omega_{\mathrm{FS}}$ holds.

Symplectic Geometry and Classical Mechanics: Exercises

University of Regensburg, Summer term 2023
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Please hand in the exercises until **Monday, June 19th** in the lecture

Exercise Sheet no. 8

Exercise 1: *conformal symplectic maps* (4 points).
Let (M_1, ω_1) and (M_2, ω_2) be a symplectic manifolds. A smooth map $f\colon M_1 \to M_2$ is called a *conformal symplectic map* if there exists a smooth function $h\colon M_1 \to \mathbb{R}$ with

$$f^*\omega_2 = h \cdot \omega_1 .$$

a) Show that if the dimension of M_1 is strictly bigger than 2, then the factor h is locally constant.

b) Find conformal symplectic maps for $M_1 = M_2 = \mathbb{R}^{2n}$ and $n > 1$, s.t. the conformal factor h is 0 or ± 1.

c) Find a conformal symplectic map of $M_1 = M_2 = S^2$ s.t. the conformal factor is non-constant.

d) *Bonus:* Let $h \in C^\infty(S^2)$ be given. Can you find a conformal symplectic map $f\colon S^2 \to S^2$ with conformal factor h?

Exercise 2 (4 points).
We consider the following map

$$\begin{aligned} \iota\colon (B_1(0), \omega_{\mathrm{std}}) &\to (\mathbb{C}P^1, \omega_{\mathrm{FS}}) \\ z &\mapsto [z : \sqrt{1-|z|^2}], \end{aligned}$$

where $B_1(0) \subset \mathbb{C}$ is the standard open ball of radius 1 in the complex plane and the induced standard symplectic form of $\mathbb{C}$. The symplectic form ω_{FS} is the Fubini-Study form introduced on Exercise sheet 7, Ex. 4.
Show that the image of ι is open and dense, and that ι is a symplectomorphism on its image.

Exercise 3: *Calibrations* (4 points).
Let $G \subset \mathbb{C}$ be open, bounded and non-empty subset of the complex plane with smooth boundary.

a) Show the following inequality for all $X, Y \in \mathbb{R}^{2n}$:

$$\omega_{\mathrm{std}}(X,Y)^2 \leq \langle X,X\rangle \langle Y,Y\rangle - \langle X,Y\rangle^2 ,$$

where ω_{std} is the standard symplectic form on $\mathbb{R}^{2n}$. Moreover, the equality case in the inequality case holds iff X, Y are linear dependent.

b) Let $F\colon \overline{G} \to \mathbb{C}^n$ be a smooth map. Show the *Wirtinger inequality*, i.e.

$$\int_G F^*\omega_{\mathrm{std}} \leq \mathrm{area}(F(G)) \coloneqq \int_G \sqrt{\langle \partial_x F, \partial_x F\rangle \langle \partial_y F, \partial_y F\rangle - \langle \partial_x F, \partial_y F\rangle^2}\, \mathrm{d}x\, \mathrm{d}y.$$

The equality case holds iff the map F is holomorphic on G.

c) Let $H : \overline{G} \times [0,1] \to \mathbb{C}^n$ be a homotopy of smooth maps which fixes the boundary of G, i.e. $H(t,x) = F_0(x)$ for all $x \in \partial G$ and $t \in [0,1]$, where we write $F_i \coloneqq H(i,\cdot)$ for $i = 0,1$. Assume that F_0 is holomorphic on G. Show that:

$$\operatorname{area}(F_1(G)) \geq \operatorname{area}(F_0(G)).$$

Hint: One can use that the homotopy H given as above satisfies $\int_G F_0^*\omega_{std} = \int_G F_1^*\omega_{std}$.

Exercise 4: *Action-angle coordinates* (4 points).
Let $E = (E_1, \dots, E_k) \in \mathbb{R}^k$. We call a Hamiltonian system (M^{2k}, ω, H) *integrable* if there exist smooth function $H_1 = H, H_2, \dots, H_k$ on M, s.t.

i) The Poisson-brackets vanish, i.e. $\{H_i, H_j\} = 0$ for all $i, j \in \{1, \dots, k\}$,

ii) For all points $x \in N_E \coloneqq \{y \mid H_i(y) = E_i \text{ for all } i \in \{1, \dots, k\}\}$ the symplectic gradients $\operatorname{sgrad} H_1, \dots, \operatorname{sgrad} H_k$ are linear independent at x.

Assume that N_E is non-empty, compact and connected and let $x_0 \in N_E$.

a) Show that the map

$$F_{x_0} \colon \mathbb{R}^k \to M, (t_1, \dots, t_k) \mapsto \Phi_{t_1}^{H_1} \circ \dots \circ \Phi_{t_k}^{H_k}(x_0)$$

is well-defined and an immersion. Moreover the image of F is given by $\operatorname{image}(F) = N_E$.

b) The preimage $F_{x_0}^{-1}(x_0)$ is a discrete and closed subgroup of $\mathbb{R}^k$, which is generated by linear independent elements $v_1 \dots, v_k \in \mathbb{R}^k$.

c) Construct a diffeomorphism $G \colon T^k \coloneqq \mathbb{R}^k/\mathbb{Z}^k \to N_E$, s.t. there exist $w_0, w_1 \in \mathbb{R}^k$ with $\Phi_t^H(x_0) = G([w_0 + tw_1])$.

d) Let $C \coloneqq \overline{\{\Phi_t^H(x_0) \mid t \in \mathbb{R}\}}$ be the trajectory of the gradient flow of H. Show that $G^{-1}(C) \subset T^k$ is a closed submanifold. Determine all possible dimensions of C for different Hamiltonian systems (M, ω, H).

Symplectic Geometry and Classical Mechanics: Exercises

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Please hand in the exercises until **Monday, June 26th** in the lecture

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Exercise Sheet no. 9

Exercise 1 (4 points).
Let (M,ω) be a symplectic manifold and recall that the Poisson bracket is given by $\{f,g\} := \omega(\operatorname{sgrad} f, \operatorname{sgrad} g)$. Show that in Darboux coordinates $(U,(q_i,p_i))$, i.e. $\omega_{|U} = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$ holds, the Poisson bracket $\{f,g\}$ can be written as

$$\sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

Exercise 2 (4 points).
We consider the following complex structures on $\mathbb{C}^2$

$$I_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and define $J := \sum_{k=1}^3 f_k I_k$ for constants $f_k \in \mathbb{R}$ with the condition $f_1^2 + f_2^2 + f_3^2 = 1$.

a) For which $(f_k)_k$ is the structure J a compatible complex structure for the symplectic form $\omega = -\mathfrak{Im}(\langle\cdot,\cdot\rangle_{\mathbb{C}^2})$?

b) For which $(f_k)_k$ is the structure J a compatible complex structure for the symplectic form $\omega = -\mathfrak{Im}(\langle I_1\cdot,\cdot\rangle_{\mathbb{C}^2})$?

Bonus: Consider the almost complex structure $J := \sum_{k=1}^3 f_k I_k$ on the symplectic manifold $(\mathbb{C}^2, \omega_{\mathrm{std}})$ with functions $f_k\colon \mathbb{C}^2 \to \mathbb{R}$ which satisfy the constraint $f_1^2 + f_2^2 + f_3^2 = 1$ on $\mathbb{C}^2$. Show that J is an integrable almost complex structure iff f_k is constant for all k.

Exercise 3 (4 points).
Show that the space of all complex structure on $\mathbb{C}^n$, which are compatible with the standard scalar product, is given by the space

$$\mathrm{O}(2n)/\,\mathrm{U}(n).$$

Show moreover, that for the case $n = 2$ this space can be identified with

$$S^2 \sqcup S^2.$$

Exercise 4 (4 points).
Consider the map

$$f\colon \mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^4 \setminus \{0\}$$
$$x \to \frac{x}{\|x\|^2}$$

and a complex linear structure $J \in \operatorname{End}(\mathbb{R}^4)$.

a) Show that the pullback $J_f \coloneqq f^*J$ is a complex structure on $\mathbb{R}^4 \setminus \{0\}$. Is there an extension of J_f to all of $\mathbb{R}^4$?

b) Let ∂_r the radial vector on $\mathbb{R}^4 \setminus \{0\}$. We decompose the space $\alpha: \mathbb{R}^4 \setminus \{0\} \cong S^3 \times \mathbb{R}_{>0}, x \mapsto (\frac{x}{\|x\|}, \|x\|)$ and thus also the tangent bundle as

$$\mathrm{d}\alpha: T_p\mathbb{R}^4 \setminus \{0\} \to T_{p/\|p\|}S^3 \oplus T_{\|p\|}\mathbb{R}_{>0}.$$

Show there exists a vector field $X \in \Gamma(TS^3)$, s.t. $(\partial_r, J\partial_r, X, JX)$ is an orthonormal frame of $T\mathbb{R}^4 \setminus \{0\}_{|S^3}$. Show moreover that $\mathrm{d}f$ can be decomposed as

$$\mathrm{d}\alpha \circ \mathrm{d}f \circ \mathrm{d}\alpha^{-1} = \mathrm{id}_{T_{\varphi p}S^3} \oplus -\mathrm{id}_{T_1\mathbb{R}_{>0}}$$

for a point $p \in S^3$.

c) Show that the commutator of the $[J, J_f] = 0$ vanishes. *Bonus: Can you generalize the argument for arbitrary linear complex structures* $J_1, J_2 \in \mathrm{End}(\mathbb{R}^4)$, *which induce different orientations on* $\mathbb{R}^4$*?*

d) We consider the restricted map J_f on the bundle $\pi^{S^3}: \mathbb{R}^4 \times S^3 \to S^3$. Show that for every point $p \in S^3$, the map J_f is constant along a Hopf circle $C_p \coloneqq S^3 \cap \mathrm{span}\{\partial_r, J_f\partial_r\}_p$.

e) Let $H: S^3 \to \mathbb{C}P^1$ be the quotient map. Let $[p] = L \in \mathbb{C}P^1$ be a complex line and define the following map:

$$\begin{aligned}\varphi: \mathbb{C}P^1 &\to S^2\\ [p] &\mapsto J(C_p),\end{aligned}$$

where $J(C_p)$ is induced complex structure on $\mathbb{R}^4$ given by $(\partial_r, J_f\partial_r, X, J_fX)$. Show that φ is a diffeomorphism.

Symplectic Geometry and Classical Mechanics: Exercises

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Exercise Sheet no. 10

Exercise 1 (4 points).
Let (M, ω, g, J) be a manifold equipped with a non-degenerated 2-form ω, a Riemannian metric g and an almost complex structure J. Assume that ω and J are compatible with $g = \omega(\cdot, J\cdot)$. Show that if J is g-parallel, i.e. $\nabla^g J = 0$, then J is integrable and ω is parallel and moreover ω is closed.

Exercise 2 (4 points).
Let $P_1, \ldots, P_k$ be homogeneous Polynomials in $(n+1)$-variables. Assume that for every point $z \in \mathbb{C}^{n+1} \smallsetminus \{0\}$ with $P_1(z) = \ldots = P_k(z) = 0$ the differentials

$$\mathrm{d}_z P_1, \ldots, \mathrm{d}_z P_k$$

are linear independent. Show that subset

$$\bigcap_{i=1}^{k} P_i^{-1}(\{0\}) \subset \mathbb{C}P^n$$

is a complex submanifold.

Exercise 3: *Segre embedding* (4 points).
Let V, W be finite dimensional complex vector spaces. The map

$$\begin{aligned} \iota_{V,W} \colon \mathbb{P}(V) \times \mathbb{P}(W) &\to \mathbb{P}(V \otimes W) \\ ([v],[w]) &\mapsto [v \otimes w] \end{aligned}$$

is called the *Segre embedding*, where we denote the projectivization of V by $\mathbb{P}(V) = V \smallsetminus \{0\}/\sim$ with the equivalence relation $\sim$ given by: Let $v, w \in V \smallsetminus \{0\}$ be equivalent $v \sim w$ if there exists $\lambda \in \mathbb{C} \smallsetminus \{0\}$ such that $v = \lambda w$. Show:

a) Let M be a complex manifold and $N \subset M$ a real submanifold and assume that the integrable complex structure J^M of M preserves the tangent bundle of N, i.e. $J^M(TN) \subset TN$, then N is a complex submanifold of M.

b) The map ι is an embedding and the image is a complex submanifold. What is the codimension of the image?

Exercise 4 (4 points).
Let M^{2n} be a complex manifold with real dimension $2n$. Recall that the complexified tangent bundle $T_{\mathbb{C}}M$ splits into the $\pm i$-Eigensubbundles of the integrable complex structure J of M, i.e. $T_{\mathbb{C}}M \coloneqq TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$. Write $\Lambda^{1,0} = T^{1,0}M$ and $\Lambda^{0,1} = T^{0,1}M$. We define $\Lambda^{p,0} \coloneqq \bigwedge_{i=1}^{p} \Lambda^{1,0}$ and respectively $\Lambda^{0,q} \coloneqq \bigwedge_{i=1}^{q} \Lambda^{0,1}$. We have the map

$$\begin{aligned} \iota \colon \Lambda^{p,0} \otimes \Lambda^{0,q} &\to \Lambda^k_{\mathbb{C}} \coloneqq \Lambda^k \otimes_{\mathbb{R}} \mathbb{C} \\ \alpha \otimes \beta &\to \alpha \wedge \beta \end{aligned}$$

and set $\Lambda^{p,q} \coloneqq \mathrm{image}(\iota)$.

a) Show that $T^*M \otimes_{\mathbb{R}} \mathbb{C} \cong (TM \otimes_{\mathbb{R}} \mathbb{C})^*$ holds.

b) Let $(\varphi: U \subset M \to V \subset \mathbb{C}^n, z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$ be a complex chart of M such that $T^*_{\mathbb{C}}M$ is trivialized over U. Show that $\mathrm{d}z_1, \ldots \mathrm{d}z_n, \mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n$ is a basis of $T^*_{\mathbb{C}}U$.

c) Construct an isomorphism $\bigoplus_{p+q=k} \Lambda^{p,q} \cong \Lambda^k_{\mathbb{C}}$.

Symplectic Geometry and Classical Mechanics: Exercises

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Exercise Sheet no. 11

Exercise 1 (4 points).
Let G be a smooth manifold and a group. Assume that the group multiplication $m\colon G\times G \to G$ is a smooth map. Show that the inversion $\iota\colon G \to G$ is also smooth.
Hint: Use the equation $m(\iota(g), g) = e$ for all $g \in G$ and e the neutral element of the group.

Exercise 2 (4 points).
Let $S_1, S_2 \in \mathbb{R}^{n\times n}$ be symmetric matrices. Define for A and B in the Lie algebra $\mathfrak{so}(n) \coloneqq \{M \in \mathbb{R}^{n\times n} \mid M^T = -M\}$ given by

$$\langle A,B\rangle_{S_1,S_2} \coloneqq \operatorname{tr}\left(S_2 A S_1 B^T\right). \tag{1}$$

a) Let S_1 and S_2 be positive definit. Show that $\langle \cdot,\cdot\rangle_{S_1,S_2}$ is a scalar product on $\mathfrak{so}(n)$.

b) Show that for any scalar product $\langle\!\langle \cdot,\cdot\rangle\!\rangle$ on $\mathfrak{so}(3)$, there exists a unique symmetric matrix $S_3 \in \mathbb{R}^{n\times n}$ with $\langle\cdot,\cdot\rangle_{S_3,\mathrm{id}} = \langle\!\langle \cdot,\cdot\rangle\!\rangle$. Is S_3 always positive definit.

c) Prove the existence of scalar products on $\mathfrak{so}(4)$ which are not of the form as in eq. (1).

d) Show that every scalar product on $\mathfrak{so}(n)$ is a sum of scalar products as in eq. (1) with positive definit S_1 and S_2.

Exercise 3 (4 points).
Let G be a Lie group. We consider the adjoint action of G on its Lie algebra:

$$\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g}), g \mapsto (X \mapsto \mathrm{d}_e l_g \circ \mathrm{d}_e r_{g^{-1}}(X))$$

where $l, r\colon G \to \mathrm{Aut}(G)$ are the left and right multiplication of the group G. Let $\langle \cdot,\cdot\rangle$ be a scalar product on $\mathfrak{g}$. We say that $\langle \cdot,\cdot\rangle$ is Ad-invariant if $\mathrm{Ad}_g^*\langle \cdot,\cdot\rangle = \langle \cdot,\cdot\rangle$ holds for any $g \in G$. A tensor (field) on G is called bi-invariant, if it is both left- and right-invariant.

a) Show: a scalar product $\langle \cdot,\cdot\rangle$ on $\mathfrak{g}$ can be extended to a bi-invariant Riemannian metric, if and only if $\langle \cdot,\cdot\rangle$ is Ad-invariant.

b) Assume that we know $\mathrm{d}_e(\mathrm{Ad})(X) = \mathrm{ad}_X = [X,\cdot]$ for all $X \in T_e g = \mathfrak{g}$. Show: if $\langle \cdot,\cdot\rangle$ is Ad-invariant, then ad_X is skew-symmetric w.r.t. $\langle \cdot,\cdot\rangle$. Is the converse true as well? Or is it true under additional assumptions?

c) Let γ be a left-invariant Riemannian metric extending the scalar product $\langle \cdot,\cdot\rangle$ on $\mathfrak{g}$. Show that ad_X is skew-symmetric w.r.t. $\langle \cdot,\cdot\rangle$, iff the Levi-Civita connection for γ is given by $\nabla^\gamma_X Y = \frac{1}{2}[X,Y]$ for all $X, Y \in \mathfrak{g}$. *Hint: Use the Koszul formula.*

d) Show that the induced Riemannian exponential function and the Lie exponential on G coincide for a biinvariant metric γ.

e) *Bonus exercise:* Let G be a connected Lie group with a left invariant metric γ. Show that if the induced Riemannian exponential map and the Lie exponential map coincide, then γ is biinvariant.

Exercise 4: *Coadjoint orbit* (4 points).
Let G be a Lie group and $\mathfrak{g}$ its Lie algebra and $\xi \in \mathfrak{g}^*$ be an element in the dual. We have the adjoint action $\mathrm{Ad}: G \to \mathrm{GL}(\mathfrak{g})$ and the induced coadjoint action Ad^* given by $(\mathrm{Ad}_g^* \xi)(X) = \xi(\mathrm{Ad}_g X)$ for $X \in \mathfrak{g}, \xi \in \mathfrak{g}^*$ and $g \in G$. Note that the coadjoint action Ad^* is a right action, i.e. $\mathrm{Ad}_{gh}^* = \mathrm{Ad}_h^* \circ \mathrm{Ad}_g^*$. Similarly we define $\mathrm{ad}_X^* \xi \in \mathfrak{g}$ by

$$(\mathrm{ad}_X^* \xi)(Y) := \xi(\mathrm{ad}_X(Y)) = \xi([X,Y]) \quad \forall Y \in \mathfrak{g} .$$

For a fixed covector $\mu \in \mathfrak{g}^*$ we define the *coadjoint orbit* $\mathcal{O}_\mu := \{\mathrm{Ad}_g^*(\mu) \mid g \in G\}$.

a) Show that the coadjoint orbit $\mathcal{O}_\mu$ is submanifold of $\mathfrak{g}^*$, whose tangent space at ν is $\{\mathrm{ad}_X^* \nu \mid X \in \mathfrak{g}\}$.
If helpful, you may use without proof, that any closed subgroup H of a Lie group G is a submanifold, and then G/H carries a unique manifold structure, such that the projection $G \to G/H$ is a submersion.
Show that Ad^* defines a smooth and transitive action of G on $\mathcal{O}_\mu$.

b) For $\nu \in \mathcal{O}_\mu$ and $X, Y \in T_\nu \mathcal{O}_\mu$ we define

$$\omega_\nu : T_\nu \mathcal{O}_\mu \times T_\nu \mathcal{O}_\mu \to \mathbb{R}, \quad \omega_\nu(\mathrm{ad}_X^* \nu, \mathrm{ad}_Y^* \nu) := \nu([X,Y]) .$$

Show that ω_ν is an alternating, non-degenerate bilinear map.

c) Show that this 2-form ω is a G-invariant symplectic form on the coadjoint orbit $\mathcal{O}_\mu$. In fact, show that ω is closed and that for all $g \in G$ we have $\left(\mathrm{Ad}_g^*\right)^* \omega = \omega$.