Seminar on semiclassical analysis

Summer term 2023

Prof. Bernd Ammann

Time and Place of the seminar: Tuesday 14.15 to 16.00, in M009

Number of sessions: **11**

Available Dates 18.4., 25.4., 2.5, 9.5., 16.5., 23.5., **16.6.**, 20.6., 27.6., 4.7., 25.7.

Special obstructions:

- June 6th: Bernd in Hamburg
- June 16th: Extra session
- July 11th: Bernd in Stockholm
- July 18th: Workshop "Gauge theory with applications to geometry and low-dimensional topology" in Regensburg (the full week July 17th–21st)
- July 25th: Extra session

Two holidays

Content

In the seminar we mainly follow Zworski's book [6]. An alternative source is [4].

The goal of the seminar is to understand semiclassical theory. In this context classical theory is classical mechanics, viewed as a Hamiltonian system on the symplectic manifold T^*M. The Hamilton function is e.g.

$$H(x,\xi) = \frac{1}{2}g(\xi,\xi) + V(x)$$

which has a kinetic term $\frac{1}{2}g(\xi,\xi)$ and a potential term V. After quantization this turns into an operator

$$\mathcal{H}_h := -h^2\Delta + V,$$

where Δ is the analyst's Laplace operator[1]. In the physical interpretation, h is Planck's constant, and motivated by physics, and the mass is normalized to be $1/2$, which often can be achieved by choosing appropriate scales. One would like to understand whether and how the geometric properties of the quantized system converge in some sense towards the geometric properties of the classical

[1] This is minus the Laplace-Beltrami operator of a geometer

system, when Planck's constant h converges to 0 or when the energy of the particle converges to ∞.

More precisely, assuming M to be closed, a solution of the quantized system is represented by an eigenfunction for $\mathcal{H}_h$, thus we assume that $u_{h,i}$, $i \in \mathbb{N}$ are the pairwise orthogonal, L^2-normalized eigenfunctions for $\mathcal{H}_h$ to the eigenvalues $\lambda_{h,i}^2$ ordered by claiming

$$\lambda_{h,1} \leq \lambda_{h,2} \leq \dots .$$

Now let A be a self-adjoint pseudo-differential operator with symbol σ_A, assuming that σ_A is integrable as a function on the unit sphere S^*M in T^*M. We ask for example whether the quantum expectation value $\int_M (Au_{h,i})u_{h,i}$ converges to the classical expectation value $\int_{S^*M} \sigma_A$ for all $i \to \infty$, $i \in \mathbb{N}$ or for some subsequence $i_\ell \to \infty$. Such problems are considered under the topic *quantum (unique) ergodicity.* The physical motivation is whether the value of the quantized observable tends to the value of the classical observable in the high energy limit, however our focus lies on an exact mathematical understanding of this.

Mathematically this leads to very challenging problems, where important progress was obtained by Nonnenmacher, Anantharaman, Lindenstrauss and many others, going farther than what we can aim for in the seminar. We refer [5] for an overview over these modern developments.

Instead the seminar's goal lies in a good understanding of the foundations.

A first step towards understanding such effects will be achieved by showing that approximate solutions of the PDE $P_h u_h = 0$ concentrate in some sense in the zero set of the associated symbol for $h \to 0$. We refer to [6, Theorem 5.3] or to Talk no. 2 for details. This has consequences for wave dumping.

The subjects of further talks will be explained below.

Prerequistites

We assume that the particpants are familiar with symplectic manifolds and the quantization procedure that associates operators to symbols. This covers roughly [6, Chapter 1–4].

Semi-classical analysis for partial differential equations and eigenfunctions

We treat Chapters 5–9 of [6]. According to Zworski, these chapters are logically independent, they mainly rely on Chapter 2–4 directly. However, Chapter 7 continues the investigation about eigenvalues and functions of Chapter 6, thus I doubt that it were easy to treat Chapter 7 without Chapter 6.

Talk no. 1: Semiclassical defect measures. *18.4. +25.4.* RAPHAEL SCHMIDPETER+*2.5 +9.5.* JONATHAN GLÖCKLE
Explain [6, Chapter 5]. For simplicity the chapter only considers the cases

$M = \mathbb{R}^{2n} = T^*\mathbb{R}^n$ and $M = T^*\mathbb{T}^n$, where $T^*\mathbb{T}^n$ is a flat n-dimensional torus, although most of the results extend to closed manifolds in general.

Assume that a sequence of L^2-bounded functions u_h, $h \in (0, h_0]$ is given. Further assume that $a \in C_c^\infty(\mathbb{R}^{2n}) = C_c^\infty(T^*\mathbb{R}^n)$ is a compactly supported symbol with Weyl quantizations $a^W(x, hD)$, again $h \in (0, h_0]$. Then Theorem 5.2 in [6] says that a sequence $h_j \to 0$ and a measure μ on $\mathbb{R}^{2n}$ exist, both not depending on a, but on the family (u_h) such that

$$\langle a^w(x, h_j D) u_{h_j}, u_{h_j} \rangle \to \int_{\mathbb{R}^{2n}} a(x, \xi)\, d\mu.$$

Such measures μ are called *semiclassical defect measures* and are the subject to the talk.

It is shown that if u_h, $h > 0$ are almost solutions of $P_h u_h = 0$, where P_h is the quantization of an elliptic operator, then the support of any defect measure is in the zero set of the symbol (Theorem 5.3). This leads to Theorem 5.4 that may be interpreted to say that the semiclassical defect measure (i.e. the location density of the quantum system) is invariant under the Hamiltonian flow (i.e. a union of classical orbits). As a conclusion, one shows that under some dynamical hypothesis, see (5.3.8), the energy of solutions to the damped wave equation with periodic coefficients on $\mathbb{R}^n$ decays exponentially. This is the statement of Theorem 5.10 which is due to Rauch and Taylor, and it provides a first classical/quantum correspondence.

Talk no. 2: A harmonic oscillator approach to Eigenvalues and Eigenfunctions. *16.5. +23.5.* Christoph Krpoun+***16.6.*** Matthias Ludewig
The subject is [6, Chapter 6]. This chapter starts with a nice treatment of the harmonic oscillator, at first in 1 dimension, then n dimensions, followed by the corresponding Weyl asymptotics (Theorem 6.3). This is in fact a special case of the operator $P(h) = -h^2\Delta + V$ where V is a suitable potential. This operator and its limit for $h \to 0$ is the main topic of the chapter. The harmonic oscillator serves both as a motivating special case (here $V(x) = x^2$), whose behavior should be generalized to arbitrary operators $P(h)$ as above, but also as an important ingredient in the proof. In fact Theorems 6.2, 6.3 and 6.5 are statements about the harmonic oscillator, while Theorems 6.4, 6.6, 6.7, 6.8 are about general such operators. Theorem 6.6 may be interpreted as follows: the values of the quantum observables (i.e. spectrum of the operator in Hilbert space) is contained in the values of the classical variables, after a perturbation in $O(h^\infty)$. Thus this is a classical–quantum relation as desired.

In Sections 6.3 and 6.4 the resolvent and Weyl's law are discussed for $P(h)$. The content of Section 6.3 consists mainly out of standard facts about elliptic operators and their resolvents, however presented in a way, that makes the h-dependence as explicit as possible. The main interest in this part lies in Section 6.4 that proves a version of Weyl's law which also sheds the h-dependence clearly. The asymptotics of the number of eigenvalues below $\Lambda \in \mathbb{R}$ is associated to the volume of subenergy subsets

$$\{(x, \xi) \in T^*\mathbb{R}^n \mid |\xi|^2 + V(x) \le \Lambda\}.$$

Talk no. 3: Tunnelling effects and L^∞-estimates for quasimodes. *20.6. +27.6. +4.7.* Guadalupe Castillo Solano+Roman Schiessl
Here we treat [6, Chapter 7]. Again, we treat the operator $P(h) = -h^2\Delta + V$. Let $u(h)$ be an eigenfunction to the eigenvalue $E(h)$. The main results of Section 7.1 (Theorems 7.3 and 7.4) state that in regions which are forbidden for the classical system (due to energy constraints) the corresponding eigenfunctions $u(h)$ decay exponentially, i.e. as $e^{-\delta/h}$ for some $\delta > 0$. This effect corresponds to what is taught in quantum mechanics lectures in view of simple examples. Theorem 7.4 can be seen as a Carleman type estimate. If time admits, one could briefly mention that such estimates are also used to prove unique continuation for some elliptic operators, for the Dirac operator see [3, Part I, Section 8] and [2] for this side aspect. Analogous literature exists for Schrödinger type operators, see e.g. original work by Aronszajn [1] and Cordes.

In Section 7.2 such exponential decay estimates are extended to tunneling effects. The remaining part of the chapter heads towards studying quasimodes which is defined as a family $u(h)$, $h \in (0, h_0]$ as above. The behaviour for $h \to \infty$ is studied and compared to the corresponding classical system. The last section (Section 7.5) replaces the bounds in terms of Sobolev spaces by bounds in Hölder spaces $C^{k,\alpha}$. It is up to the speaker(s) to decide, whether (s)he wants to treat this alternative version as well, whether (s)he wants only to sketch some main ideas, or whether it should be skipped completely.

In Section 7.5 the book presents a self-contained "semiclassical" derivation of interior Schauder estimates for Laplace type operators.

Propagation of singularities on manifolds in the semiclassical context

Talk no. 4: More on the symbol calculus. *25.7.* Julian Seipel
We discuss [6, Chapter 8].

The first main result of this talk is Beal's theorem (Theorem 8.3), which says that an h-dependent linear operator $A : \mathcal{S} \to \mathcal{S}'$ is the operator associated to a symbol, if and only if any N-th iterated commutator of linear functionals wirth A vanish of order h^N.

Section 8.2 treats well-defined symbols of the form $g := \log m$, where M is a symbol in the previous sense. The corresponding operators may be exponentiated, and the properties of this exponential is the main topic here. This leads to a generalization of Sobolev spaces in Section 8.3.

In Section 8.4 the wavefront set is introduced, and then one studies the wavefront set for symbols. Under diffeomorphisms, the wavefront set behaves as a subset of the cotangent bundle. This leads to microlocality.

—END OF THE SEMINAR—

As discussed above, so far all Chapters are independent, according to the author. Now, Chapters 9–12 continue along the lines of Chapter 8. Chapter 9 should be known from the previous semester, but the following talks would be interesting, if time admits. The chapters 13 and following seem out of reach in the summer term.

Supplementary Talk no. 1: Symbols, Ψdo's and Beal's theorem on manifolds N.N.

[6, Chapter 9]. This talk adapts the definition of symbols and pseudodifferential operators to manifolds. We probably already the essential parts of this chapter in the winter term in sufficient detail. Therefore it might be wiser to concentrate on other chapters, and to skip this talk.

Talk no. 5: Fourier integral operators. *next term* **(or will be skipped)** N.N.

We treat [6, Chapter 10].

Let p_t be a family of real-valued symbols, $t \in \mathbb{R}$; and $P(t)$ its Weyl quantization. A central topic in the chapter is to study solutions of

$$\frac{h}{i}\frac{d}{dt}F(t) + P(t)F(t) = 0 \tag{1}$$

where $F(t)$ is a family of operators, with $F(0) = \mathrm{Id}$. As $P(t)$ is self-adjoint, the family $F(t)$ consists of unitaries. If $P(t)$ is the Hamiltonian, this equation is the time evolution of quantum mechanics in the Schrödinger picture.

Solutions of (1) will be given by an integral formula, using oscillatory integrals, which provides the WKB approximation, named after Wentzel, Kramers and Brillouin. For providing such a representation interesting methods of proof are used, e.g. the Hamilton-Jacobi equation.

This integral formula is then be applied in Section 10.3 in order to obtain L^p-bounds for quasimodes. So-called *Strichartz estimates* are a first step in this direction (Section 10.3), and finally the L^p-bounds are obtained in Section 10.4.

Talk no. 6: Egorov's theorem and variants. *next term* **(pushed to later)** N.N.

We now discuss [6, Chapter 11]. Let $A = a^W(x, hD)$ be a pseudo-differential operator, representing an observable whose evolution shall be discussed in the Heisenberg picture. For this we set $A(t) := f(t)^{-1}AF(t)$. From (1) we get the evolution of A in the Heisenberg picture

$$\frac{d}{dt}A(t) = \frac{i}{h}[Q(t), A(t)].$$

Natural question arise: is $A(t)$ again a pseudo-differential operator, more pricisely of the form $A(t) = b_t^W(x, hD)$ for some family of symbols b_t^W? This is answered by Egorov's Theorem (Theorem 11.1). It also shows that up to terms $O(h)$ the symbol b_t^W coincides with the pullback of a^W by the 1-parameter family of symplectic diffeomorphisms generated by the Hamiltonian vector field ddefined by the symbol of $P(t)$. Thus classical evolution coincides with quantum evolution (in the Heisenberg picture) up to terms $O(h)$. So far, this is dealt with in Section 11.1 which is the most important part.

Other sections in this chapter discuss several modifications and extensions of Egorov's theorem. Symplectic mappings are "quantized" and a microlocal version of Egorov's theorem is given. This is particular strong for linear symplectic maps. Finally in Section 11.4 Egorov's Theorem is extended to a longer time intervall; in physical terms expressed, it is valid up to times comparable to *Ehrenfest time* $\log(h^{-1})$.

Supplementary Talk no. 2: More on propagation of singularities, quasimodes and pseudospectra *next term* N.N.

If time admits, we then will discuss [6, Chapter 12]. A main result is [6, Theorem 12.5] which gives a refinement about the evolution of the wavefront set (already discussed in [6, Theorem 5.4]. This is helpful for a better understanding of solution of both hyperbolic equations (wave equations) and parabolic equations (heat equations). A more conceptual framework is provided by quasimodes and pseudospectra.

More details might be given later, if we have time for this talk.

Seminar-Homepage

`https://ammann.app.uni-regensburg.de/lehre/2023s_semiclassic`

Literatur

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[3] Booss Bavnbek, B., and Wojciechowski, K. P. *Elliptic boundary problems for Dirac operators.* Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1993.

[4] Dimassi, M., and Sjöstrand, J. *Spectral asymptotics in the semi-classical limit*, vol. 268 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1999.

[5] Zelditch, S. Recent developments in mathematical quantum chaos. In *Current developments in mathematics, 2009.* Int. Press, Somerville, MA, 2010, pp. 115–204. arXiv: 0911.4312.

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