Geometric PDEs on Manifolds: Exercises

University of Regensburg, Winter term 2022/23
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Please hand in the exercises until **Tuesday, February 7th**

Exercise Sheet no. 14

Exercise 1 (4 points).
Let (M, g) be a Riemannian manifold of dimension $n \geq 3$, $p \in M$ and $N \in \mathbb{N}$. Show that there is a metric $\overline{g} \in [g]$ for which

$$\mathrm{Sym}(\overline{\nabla}^k \operatorname{ric}^{\overline{g}})_{|p} = 0$$

holds for all $k = 0, \ldots N$.

Exercise 2 (4 points).
Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $\mathrm{scal}^g \geq s_0$ for a positive constant $s_0 > 0$. Let $u \in C^\infty(M)$ be a minimizer of the Yamabe functional with $\|u\|_{L^p} = 1$. The aim of this exercise is to show that

$$\|u\|_{L^\infty} \geq \left(\frac{s_0}{\lambda(\mathbb{S}^n)} \right)^{\frac{n-2}{4}}. \tag{1}$$

You may proceed in the following steps:

a) Find a lower bound for

$$Q_2^g(v) = \frac{\int_M v Y^g v \,\mathrm{dvol}}{\|v\|_{L^2}^2}$$

that holds independently of $v \in H^{1,2}(M) \setminus \{0\}$.

b) Estimate $Q_2^g(u)$ from above in terms of $\|u\|_{L^\infty}$.

c) Conlude (1).

Exercise 3 (4 points).
Let $n \in \mathbb{N}$ and $\sigma \colon S^n \setminus \{e_0\} \to \mathbb{R}^n$ be the stereographic projection. Recall that in Exercise 4 on sheet 10 we constructed a map $O(n+1, 1) \to \mathrm{Conf}(S^n), A \mapsto \tilde{A}$.

a) Show that there is an $A \in O(n+1, 1)$ such that $\sigma \circ \tilde{A} \circ \sigma^{-1} \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is the reflection at the unit circle $x \mapsto \frac{x}{\|x\|^2}$.

b) Consider the conformal diffeomorphism $\Psi_\alpha \in \mathrm{Conf}(S^n)$ induced by the dilation map $\delta_\alpha(x) = \alpha x$, $\alpha > 0$, from Exercise 1 on sheet 11. Show that $\Psi_\alpha = \tilde{A}_t$ for a Lorentz boost

$$A_t = \begin{pmatrix} \cosh(t) & \sinh(t) & \\ \sinh(t) & \cosh(t) & \\ & & \mathrm{id}_{\mathbb{R}^n} \end{pmatrix}$$

and determine t in terms of α.

Exercise 4 (4 points).
Consider the Riemannian product $(M, g) = (S^k \times H^l, g_{\text{sph}} + g_{\text{hyp}})$ for $k \geq 2$ and $n = k + l \geq 3$. Let $u \in L^p(M)$ be a positive function, which satisfies $\|u\|_{L^p} = 1$ and $Y^g u = \lambda u^{p-1}$ in the weak sense, for $p = \frac{2n}{n-2}$.

a) Construct a positive weak L^p-solution of $Y^{\tilde{g}} v = \lambda v^{p-1}$ on $(S^n \setminus S^{n-k-1}, \tilde{g} = g_{\text{sph}})$ with $\|v\|_{L^p} = 1$.

b) Show that v extends to a smooth solution on (S^n, g_{sph}) and conclude $\lambda = n(n-1)\omega_n^{\frac{2}{n}}$.