Geometric PDEs on Manifolds: Exercises

University of Regensburg, Winter term 2022/23
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Please hand in the exercises until **Tuesday, January 31th**

Exercise Sheet no. 13

Exercise 1 (4 points).
Let $n \in \mathbb{N}$ and $B \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that there is a positive constant γ_n depending only on n such that

$$\int_{S^{n-1}} \langle Bx, x\rangle \,\mathrm{dvol} = \gamma_n \operatorname{tr}(B)$$

and determine its value.

Exercise 2 (4 points).
A Riemannian manifold (M, g) is *conformally flat* if for every $x \in M$ there is an open neighborhood of x that is conformally equivalent to an open subset of Euclidean space, see Definition 2.15 in the script.

a) Let (M, g) be conformally flat of dimension $n \geq 3$ and $x \in M$. Show that there is some $\rho > 0$ such that every function $f \in C^\infty(M) \setminus \{0\}$ with compact support in $B_\rho(x)$ satisfies $Q^g(f) \geq \lambda(\mathbb{S}^n)$.

b) Let (M, g) either be the torus T^n or projective space $\mathbb{R}P^n$, for $n \geq 3$, equipped with their respective standard metric (which is conformally flat in both cases). Show that $\lambda(M, [g]) < \lambda(\mathbb{S}^n)$.

Exercise 3 (4 points).
Let G be a compact Lie group and M a compact manifold of dimension $n \geq 3$ with a smooth G-action $G \times M \to M$. For a G-invariant metric g on M, we define

$$[g]^G \coloneqq \{\tilde{g} \in [g] \mid \tilde{g} \text{ is } G\text{-invariant}\}.$$

Show that $\tilde{g} \in [g]^G$ is a stationary point of the restricted Yamabe functional $Q\colon [g]^G \to \mathbb{R}$ if and only if $\tilde{g}$ has constant scalar curvature.

Exercise 4 (4 points).
Let (M, g) be a compact $(n-1)$-dimensional Riemannian manifold with $n \geq 3$. Let $S^1 = \{\exp(it) \mid t \in \mathbb{R}\} \subset \mathbb{C}$. We consider $M \times S^1$ with the S^1-action given by multiplication in the second component. An S^1-invariant metric on $M \times S^1$ is given by $h = g + L^2\,\mathrm{d}t^2$ for $L > 0$. In analogy with Exercise 3, we consider the Yamabe functional restricted to S^1-invariant functions

$$Q^h\colon H^{1,2}(M\times S^1)^{S^1} \setminus \{0\} \to \mathbb{R}, \quad u \mapsto \frac{\int_{M\times S^1} \left(a_n |\mathrm{d}u|_h^2 + \mathrm{scal}^h u^2\right) \mathrm{dvol}^h}{\|u\|^2_{L^{p_n}(M\times S^1)}}$$

with $p_n = \frac{2n}{n-2}$ and $a_n = 4\frac{n-1}{n-2}$.

a) Show that there is a positive function $u \in C^\infty(M\times S^1)^{S^1}$ that minimizes the restricted Yamabe functional.
Hint: Check that $p_n < p_{n-1}$ and rewrite $Q^h(u)$ so that integration is performed over (M, g) only. You should be able to see that the problem of finding minimizers in the S^1-invariant setting is subcritical (and hence considerably easier than in the usual Yamabe problem).

b) Assume now that g has positive scalar curvature. Prove that there is a constant $C > 0$ such that for any $u \in H^{1,2}(M \times S^1)^{S^1} \setminus \{0\}$

$$Q^h(u) \geq \frac{\min(a_n, \mathrm{scal}^g)}{C^2} L^{2/n}.$$

c) Conclude that in the situation of part b) and for large L, the minimizer from part a) is not a minimizer of Q^h within the whole conformal class $[h]$. Conclude further that within the conformal class $[h]$ there are at least two non-isometric constant scalar curvature metrics of volume 1.