Geometric PDEs on Manifolds: Exercises

University of Regensburg, Winter term 2022/23
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Please hand in the exercises until **Tuesday, January 24th**

Exercise Sheet no. 12

Exercise 1 (4 points).
For $n \geq 3$ and $\alpha > 0$ consider the function $u_\alpha \in C^\infty(\mathbb{R}^n)$ defined by $u_\alpha(x) = \left(\frac{2\alpha}{\|x\|^2+\alpha^2}\right)^{\frac{n-2}{2}}$. Show that $\|\nabla u_\alpha\|^2, u_\alpha \Delta u_\alpha \in L^1(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \|\nabla u_\alpha\|^2 \mathrm{d}x = \int_{\mathbb{R}^n} u_\alpha \Delta u_\alpha \mathrm{d}x.$$

Exercise 2 (4 points).
Let $u \in C^\infty(\mathbb{R}^n)$ be a solution of $a\Delta u = \lambda u^{\frac{n+2}{n-2}}$ for $n \geq 3$, $a = 4\frac{n-1}{n-2}$ and $\lambda \in \mathbb{R}$. Assume that $u > 0$ and $u \in L^p(\mathbb{R}^n)$ for $p = \frac{2n}{n-2}$. Let furthermore $\sigma: S^n \setminus \{e_0\} \to \mathbb{R}^n$ be the stereographic projection and $u_1 \in C^\infty(\mathbb{R}^n)$ be defined as in Exercise 1.

a) Show that $\frac{u}{u_1} \circ \sigma: S^n \setminus \{e_0\} \to \mathbb{R}$ extends to a smooth, positive solution $v \in C^\infty(S^n)$ of $Yv = \lambda v^{\frac{n+2}{n-2}}$, where Y is the Yamabe operator of the standard round metric g_{sph}.

b) Calculate the scalar curvature of $\tilde{g} = v^{p-2} g_{\mathrm{sph}}$.

c) Show that $\tilde{g}$ has constant sectional curvature and determine its value.
Hint: Theorem of Obata.

d) Calculate $\lambda \left(\int_{\mathbb{R}^n} u^p \,\mathrm{dvol}\right)^{\frac{2}{n}}$.
Hint: Start by showing that $\int_{S^n} v^p \,\mathrm{dvol}^{g_{\mathrm{sph}}} = \mathrm{vol}(S^n, \tilde{g}) = r^n \omega_n$ for a suitable $r > 0$ and $\omega_n = \mathrm{vol}(S^n, g_{\mathrm{sph}})$.

Exercise 3 (4 points).
Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and $(s_i)_{i \in \mathbb{N}}$ be a sequence in $[2, p)$ for $p = \frac{2n}{n-2}$. Suppose that $f_{s_i} \in L^{s_i}(M)$ is a positive solution of $Y f_{s_i} = \lambda_{s_i}(M, g) f_{s_i}^{s_i - 1}$ for each $i \in \mathbb{N}$, normalized by $\|f_{s_i}\|_{L^{s_i}} = 1$. Show that if $\|f_{s_i}\|_{L^\infty} \to \infty$, then $s_i \to p$ for $i \to \infty$.
Hint: If s_i subconverges to $s_\infty < p$, find an interval $s_\infty \in (r_0, s_0)$ with $r_0 > \frac{n}{2}(s_0 - 2)$. Then make use of Theorem 3.13 (iii) from the lecture.

Exercise 4 (4 points).
We consider the embedding $S^k \subset S^n$ for $0 \leq k < n$ induced by restricting $\mathbb{R}^{k+1} \times \{0\} \subset \mathbb{R}^{n+1}$ to the unit sphere. Let $s: S^n \to \mathbb{R}$ be the (intrinsic) distance function from S^k.

a) Show that $\mathrm{im}(s) = [0, \frac{\pi}{2}]$ with $s^{-1}(0) = S^k$ and $(S^k)^\perp := s^{-1}(\frac{\pi}{2}) \cong S^{n-k-1}$.

b) Construct a diffeomorphism $\Phi: S^k \times (0, \frac{\pi}{2}) \times S^{n-k-1} \to S^n \setminus (S^k \cup (S^k)^\perp)$ such that $\mathrm{pr}_{(0,\frac{\pi}{2})} \circ \Phi^{-1} = s$ and

$$\Phi^* g_{S^n} = (\cos s)^2 g_{S^k} + \mathrm{d}s^2 + (\sin s)^2 g_{S^{n-k-1}}.$$

c) Show that precomposition with $(0, \infty) \to (0, \frac{\pi}{2}), t \mapsto \arcsin((\cosh t)^{-1}$ yields a diffeomorphism Ψ such that

$$\Psi^*((\sin s)^{-2} g_{S^n}) = (\sinh t)^2 g_{S^k} + \mathrm{d}t^2 + g_{S^{n-k-1}}.$$

d) Conclude that $(S^n \setminus (S^k \cup (S^k)^\perp), (\sin s)^{-2} g_{S^n})$ is isometric to $((H^{k+1} \setminus \{p\}) \times S^{n-k-1}, g_{H^{k+1}} + g_{S^{n-k-1}})$, where $(H^{k+1}, g_{H^{k+1}})$ is standard hyperbolic $(k+1)$-space and $p \in H^{k+1}$.

e) Prove that this isometry extends to $S^n \setminus S^k$ and $H^{k+1} \times S^{n-k-1}$, showing that $(S^n \setminus S^k, g_{S^n})$ and $(H^{k+1} \times S^{n-k-1}, g_{H^{k+1}} + g_{S^{n-k-1}})$ are conformal.