Exercise Sheet no. 11

Exercise 1 (4 points).
Consider the stereographic projections $\sigma\colon S^n \setminus \{e_0\} \to \mathbb{R}^n$ and $\sigma'\colon S^n \setminus \{-e_0\} \to \mathbb{R}^n$ as well as the dilatation and translation maps

$$\delta_\alpha\colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto \alpha x \qquad\qquad F_v\colon \mathbb{R}^n \to \mathbb{R}^n, \quad x \mapsto x + v$$

for $\alpha > 0$ and $v \in \mathbb{R}^n$.

a) Verify that $\sigma' \circ \sigma^{-1}\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is the reflection at the unit sphere, i. e. $x \mapsto \frac{x}{\|x\|^2}$.

b) Show that $\sigma^{-1} \circ \delta_\alpha \circ \sigma$ extends to a smooth diffeomorphism Ψ_α on S^n with $\sigma' \circ \Psi_\alpha \circ (\sigma')^{-1} = \delta_{\alpha^{-1}}$ and calculate $\mathrm{d}_{e_0}\Psi_\alpha$.

c) Show that $\sigma^{-1} \circ F_v \circ \sigma$ extends to a smooth diffeomorphism Φ_v on S^n and calculate $\mathrm{d}_{e_0}\Phi_v$.

Exercise 2 (4 points).
Let $n \geq 3$, $\sigma\colon S^n \setminus \{e_0\} \to \mathbb{R}^n$ be the stereographic projection and $u_1 \in C^\infty(\mathbb{R}^n)$ be the function given by

$$u_1(x) = \left(\frac{2}{\|x\|^2 + 1}\right)^{\frac{n-2}{2}}.$$

For a function $u \in C^\infty(\mathbb{R}^n)$, we set $v \coloneqq \frac{u}{u_1} \circ \sigma \in C^\infty(S^n \setminus \{e_0\})$. Assume that u is positive with

$$\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}}\,\mathrm{dvol} < \infty \qquad \text{and} \qquad 4\tfrac{n-1}{n-2}\Delta u = \lambda u^{\frac{n+2}{n-2}}$$

for some $\lambda \in \mathbb{R}$. Show that on $S^n \setminus \{e_0\}$, the function v is in L^p for $p = \frac{2n}{n-2}$ and $Yv = \lambda v^{\frac{n+2}{n-2}}$, where Y is the Yamabe operator for the standard round metric.

Exercise 3 (4 points).
Let v be a weak L^p-solution of $Yv = \lambda v^{\frac{n+2}{n-2}}$ on $S^n \setminus \{e_0\}$, where $n \geq 3$, $p = \frac{2n}{n-2}$, $\lambda \in \mathbb{R}$ and Y is the Yamabe operator of the standard round metric. Assuming $v \geq 0$, show the following:

a) v extends to a weak L^p-solution of $Yv = \lambda v^{\frac{n+2}{n-2}}$ on S^n.

b) $v \in L^r(S^n)$ for some $r > p$.

c) v is represented by a smooth function and $Yv = \lambda v^{\frac{n+2}{n-2}}$ holds classically on S^n.

Hint: Try to find an appropriate statement for each step in the lecture notes. You may also use the Lemma by Trudinger in Section 1.10 not treated so far.

Exercise 4 (4 points).
Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ with the property that the sign (considered as element of $\{-1, 0, 1\}$) of its scalar curvature is constant. Assume that $f \in C^\infty(M)$ is a positive function with $Y_g(f) = \lambda f^{\frac{n+2}{n-2}}$ for some $\lambda \in \mathbb{R}$, where $Y_g = 4\frac{n-1}{n-2}\Delta_g + \mathrm{scal}^g$ is the Yamabe operator.

a) Show that $\mathrm{sgn}(\mathrm{scal}) = \mathrm{sgn}(\lambda)$.
Hint: Look at the equation in a global maximum and minimum of f on M.

b) Assuming that M is connected, scal is constant and $\lambda \leq 0$, show that f is constant.