Geometric PDEs on Manifolds: Exercises

University of Regensburg, Winter term 2022/23
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Please hand in the exercises until **Tuesday, January 10th**

Exercise Sheet no. 10

Exercise 1 (4 points).
Minkowski space $(\mathbb{R}^{n,1}, \langle\!\langle -,- \rangle\!\rangle)$ is the vector space $\mathbb{R}^{n+1}$ with the inner product $\langle\!\langle X, Y \rangle\!\rangle = -X_0Y_0 + X_1Y_1 + \cdots + X_nY_n$ for $X = (X_0, X_1, \ldots, X_n), Y = (Y_0, Y_1, \ldots, Y_n) \in \mathbb{R}^{n+1}$. Consider the submanifold $H \coloneqq \{X \in \mathbb{R}_{>0} \times \mathbb{R}^n \mid \langle\!\langle X, X \rangle\!\rangle = -1\} \subset \mathbb{R}^{n,1}$ with its induced symmetric bilinear form g.

a) Show that g is a Riemannian metric on H and that (H, g) has constant sectional curvature $K = -1$.

b) Prove that the stereographic projection

$$\Phi\colon H \longrightarrow B_1(0) \subset \mathbb{R}^n$$
$$\mathbb{R} \times \mathbb{R}^n \ni (x_0, \vec{x}) \longmapsto \frac{\vec{x}}{x_0 + 1}$$

defines an isometry between (H, g) and $(B_1(0), g_{\mathrm{hyp}})$, where $g_{\mathrm{hyp}}(y) = \frac{4}{(1-\|y\|^2)^2} g_{\mathrm{eucl}}(y)$ for $y \in B_1(0)$ and the euclidean metric g_{eucl} on $B_1(0)$.

Exercise 2 (4 points).
Let (M, g_κ) be a Riemannian manifold of constant sectional curvature $\kappa \in \mathbb{R}$. Consider the exponential map $\exp_p\colon T_pM \supset V \to U \subset M$ in some point $p \in M$, with V star-shaped with respect to 0. After identifying S^{n-1} with the unit sphere in T_pM, we define polar coordinates by $\Phi\colon \mathbb{R} \times S^{n-1} \supset W \xrightarrow{\cong} V \setminus \{0\}$, $(r, X) \mapsto rX$. Show that $(\exp_p \circ \Phi)^* g_\kappa = \mathrm{d}r^2 + s_\kappa(r)^2 g_{\mathrm{std}}$, with g_{std} being the standard round metric on S^{n-1}, and determine the function s_κ.
Hint: Start by expressing the Jacobi field $J_W(t) = \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \exp_p(t(X + sW))$, $X \perp W$, along $c(t) = \exp_p(tX)$ in an orthonormal frame $E_1, \ldots, E_n$ that is parallel along c.

Exercise 3 (4 points).
Let (M, g_κ) and $(\tilde{M}, \tilde{g}_{\tilde{\kappa}})$ be Riemannian manifolds of constant sectional curvature κ and $\tilde{\kappa} \in \mathbb{R}$, respectively. For $p \in M$ and $\tilde{p} \in \tilde{M}$, show that there are reals $r, \tilde{r} > 0$ and a conformal diffeomorphism $\Phi\colon \tilde{M} \supset B_r(p) \to B_{\tilde{r}}(\tilde{p}) \subset \tilde{M}$. Is there a relation between r and $\tilde{r}$?

Exercise 4 (4 points).
Let $\mathcal{L} \coloneqq \{X \in \mathbb{R}^{n+1,1} \setminus \{0\} \mid \langle\!\langle X, X \rangle\!\rangle = 0\}$ be the set of light-like vectors in Minkowski space.

a) Show that $\mathcal{L}$ is a smooth (but not semi-Riemannian!) submanifold of $\mathbb{R}^{n+1,1}$ and $\iota\colon S^n \to \mathcal{L}$, $x \mapsto (1, x)$ is an embedding of smooth manifolds[1]. Prove furthermore that $S^n \xrightarrow{\iota} \mathcal{L} \to \mathcal{L}/\mathbb{R}^*$ is a bijection, giving rise to a smooth map $\pi\colon \mathcal{L} \to \mathcal{L}/\mathbb{R}^* \cong S^n$.

b) Let $x \in \mathcal{L}$. A space-like vector $v \in T_x\mathcal{L}$ defines an oriented plane $E_v = \mathrm{span}(x, v)$ in $\mathbb{R}^{n+1,1}$ ((x, v) shall be positively oriented in E_v, say). Given two oriented planes E

[1] including being a homeomorphism onto its image

and F such that $E = E_v$ and $F = E_w$ for some space-like $v, w \in T_x\mathcal{L}$, show that their enclosed angle

$$\angle(E, F) = \arccos \frac{\langle\!\langle v, w \rangle\!\rangle}{\sqrt{\langle\!\langle v, v \rangle\!\rangle}\sqrt{\langle\!\langle w, w \rangle\!\rangle}} \in [0, \pi]$$

is well-defined (i. e. independent of the choices of v and w). Verify additionally that $\angle(E, F) = \angle(\mathrm{d}_x\pi(v), \mathrm{d}_x\pi(w))$, where the latter is the angle between two vectors on the standard sphere (S^n, g).

c) Let $A \in O(n+1, 1)$. Show that A restricts to a diffeomorphism $A\colon \mathcal{L} \to \mathcal{L}$, that there is a unique map $\widetilde{A}$ making the diagram

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{A} & \mathcal{L} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S^n & \xrightarrow{\widetilde{A}} & S^n \end{array}$$

commute and that $\widetilde{A}$ is a conformal diffeomorphism.