Exercise Sheet no. 9

Exercise 1 (4 points).
Consider a Riemannian product $(M, g) = (M_1 \times M_2, \mathrm{pr}_1^* g_1 + \mathrm{pr}_2^* g_2)$ of two Riemannian manifolds (M_1, g_1) and (M_2, g_2). Calculate its Riemann curvature tensor as well as its Ricci and scalar curvature in terms of the curvatures of the factors.
Hint: Start by determining $\nabla^g_{X_i} Y_j$ for vector fields $X_i, Y_j \in \Gamma(TM)$ with $X_i(p_1, p_2) = \hat{X}_i(p_i) \in T_{p_i} M_i \subset T_{(p_1,p_2)} M$ etc. for $\hat{X}_i \in \Gamma(TM_i)$, $\hat{Y}_j \in \Gamma(TM_j)$ and $i, j = 1, 2$.

Exercise 2 (4 points).
Let $M = Q \times (T^k \setminus \{p\})$ and $\hat{M} = Q \times T^k$ be Riemannian products, where Q is a compact Riemannian manifold, T^k is the k-dimensional standard torus for some $k \geq 2$ and $p \in T^k$. Assume that $u \in L^q(M)$ is a weak solution of $\Delta u = \rho$ on M, where $\rho \in C^\infty(\hat{M})$. Determine the values of $q \in [1, \infty]$ for which this implies that u is the restriction of a function in $C^\infty(\hat{M})$, satisfying $\Delta u = \rho$ in the classical sense on all of $\hat{M}$. When this is the case, please provide a proof. When this is not the case, provide a counterexample of functions $u \in L^q(M)$ and $\rho \in C^\infty(\hat{M})$ as above, where u does not extend to a smooth function on $\hat{M}$.
Hint: Consider functions $u(x) = d_Q(x)^\alpha \chi(d_Q(x))$, where d_Q is the distance from $Q \times \{p\}$, $\alpha \in \mathbb{R}$ and χ is a suitable cut-off function.

Exercise 3 (4 points).
Let (M, g) be a compact Riemannian manifold and h a smooth function on M.

a) Let $c < \min_{p \in M} h(p)$. Prove that $\Delta + h - c \colon C^\infty(M) \to C^\infty(M)$ is invertible and that for any $k, l \in \mathbb{Z}$ (they may be negative!) there is a unique Hilbert space isomorphism $(\Delta + h - c)^k \colon H^{2l}(M) \to H^{2l-2k}(M)$ extending $(\Delta + h - c)^k \colon C^\infty(M) \to C^\infty(M)$.

b) Let $u \in H^{-\infty}(M)$ and $k \in \mathbb{Z}$. Show that $u \in H^{2k}(M)$ if and only if there is a sequence $(u_j)_{j \in \mathbb{N}}$ of eigenfunctions of $\Delta + h$ associated to distinct eigenvalues $(\lambda_j)_{j \in \mathbb{N}_0}$ such that

$$\sum_{j=0}^{\infty} (\lambda_j^2 + 1)^k \|u_j\|^2_{L^2(M)} < \infty \qquad \text{and} \qquad \sum_{j=0}^{n} u_j \overset{n \to \infty}{\longrightarrow} u \quad \text{in } H^{-\infty}(M).$$

Exercise 4 (4 points).
Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimensions n_1 and n_2, respectively, and $(M, g) = (M_1 \times M_2, \mathrm{pr}_1^* g_1 + \mathrm{pr}_2^* g_2)$ be their Riemannian product of dimension $n = n_1 + n_2$. We denote by K, R, W, ric, ric^0 and scal the sectional, Riemann, Weyl, Ricci, trace-free Ricci[1] and scalar curvature of (M, g), respectively, and add a subscript i when referring to the respective curvature of (M_i, g_i) for $i = 1, 2$.

a) Show that in general $\mathrm{ric}^0 \neq \mathrm{pr}_1^* \mathrm{ric}_1^0 + \mathrm{pr}_2^* \mathrm{ric}_2^0$ and give a criterion for equality.

b) Verify the following formula for the Weyl curvature:

$$W = R - \frac{1}{n-2} \mathrm{ric} \owedge g + \frac{1}{2(n-1)(n-2)} \mathrm{scal}\, g \owedge g.$$

[1] In the script we used B instead of ric^0 for the trace-free Ricci curvature

c) Conclude that $W \equiv 0$ if (M_1, g_1) and (M_2, g_2) have constant sectional curvature and $K_1 = -K_2$.

d) Show the converse: If $W \equiv 0$, then (M_1, g_1) and (M_2, g_2) have constant sectional curvature and $K_1 = -K_2$.
Hint: Start by looking at $W(X_1, Y_2, Z_2, \hat{X}_1)$, where $X_1, \hat{X}_1 \in TM_1 \subset TM$ and $Y_2, Z_2 \in TM_2 \subset TM$, and deduce $\mathrm{ric}_1^0 = \mathrm{ric}_2^0 = 0$ as well as $\left(\frac{1}{n_1} - \frac{1}{n-1}\right) \mathrm{scal}_1 + \left(\frac{1}{n_2} - \frac{1}{n-1}\right) \mathrm{scal}_2 = 0$.