Geometric PDEs on Manifolds: Exercises

University of Regensburg, Winter term 2022/23
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Please hand in the exercises until **Tuesday, December 13th**

Exercise Sheet no. 8

Exercise 1 (4 points).
Consider a Riemannian manifold (M, g) and a vector bundle $V \to M$ with fiberwise scalar product and compatible connection ∇. The connection defines an order one differential operator $\nabla\colon \Gamma(V) \to \Gamma(T^*M \otimes V)$

a) Show that the formal adjoint of ∇ is given by $\nabla^* = -(\mathrm{tr}^g \otimes \mathrm{id}_V) \circ \nabla^{T^*M\otimes V}$, where $\nabla^{T^*M\otimes V}$ is the connection on $T^*M \otimes V$ induced by the Levi-Civita connection on T^*M and the connection on V.

b) In the case $V = \bigwedge^k T^*M$ with induced metric and Levi-Civita connection, we define the connection Laplacian by $\Delta_k = \nabla^*\nabla\colon \Omega^k(M) \to \Omega^k(M)$. Show that $\Delta_1 \circ \nabla - \nabla \circ \Delta_0\colon C^\infty(M) \to \Omega^1(M)$ is a differential operator of order ≤ 2.

Exercise 2 (4 points).
Let (M, g) be a Riemannian manifold of dimension $n > 2$. Inside $(T^*M)^{\otimes 4} \to M$, we consider the subbundle of formal curvature tensors $\mathcal{K}(TM) \to M$, consisting of those $R \in (T_p^*M)^{\otimes 4}$, for any $p \in M$, with

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Z,W,X,Y) = -R(Y,Z,X,W) - R(Z,X,Y,W)$$

for all $X, Y, Z, W \in T_pM$. This bundle carries a metric induced by g. We define the *Kulkarni-Nomizu product*[1]

$$\begin{aligned}(T^*M)^{\odot 2} \odot (T^*M)^{\odot 2} &\to \mathcal{K}(TM)\\ h \odot k &\mapsto h \owedge k,\end{aligned}$$

where for $h, k \in (T_p^*M)^{\odot 2}$ and all $X, Y, Z, W \in T_pM$

$$\begin{aligned}h \owedge k(X,Y,Z,W) \coloneqq{}& h(X,W)k(Y,Z) + h(Y,Z)k(X,W)\\ &- h(X,Z)k(Y,W) - h(Y,W)k(X,Z).\end{aligned}$$

Consider the linear bundle maps

$$\begin{aligned}\pi^{\mathrm{scal}}\colon \mathcal{K}(TM) &\to \mathbb{R} & \pi^{\mathrm{ric}}\colon \ker(\pi^{\mathrm{scal}}) &\to (T^*M^{\odot 2})_0\\ R &\mapsto \sum_{i,j=1\ldots n} R(e_i, e_j, e_j, e_i) & R &\mapsto \sum_{i=1}^n R(e_i, -, -, e_i),\end{aligned}$$

where $e_1, \ldots, e_n$ is any orthonormal basis of T_pM for suitable $p \in M$ and $(T^*M^{\odot 2})_0$ is the bundle of trace-free symmetric 2-tensors. Let us furthermore define $\mathcal{K}_0(TM) \coloneqq \ker(\pi^{\mathrm{scal}})$ and $\mathcal{W}(TM) \coloneqq \ker(\pi^{\mathrm{ric}})$.

a) Show that the orthogonal complement of $\mathcal{K}_0(TM)$ in $\mathcal{K}(TM)$ is spanned by $g \owedge g$ and that the restriction $\pi^{\mathrm{scal}}\colon \mathcal{K}_0(TM)^\perp \to \mathbb{R}$ is an isomorphism.

[1] Convince yourself that this is well-defined!

b) Show that the orthogonal complement of $\mathcal{W}(TM)$ in $\mathcal{K}_0(TM)$ is spanned by $\{\mathrm{ric} \owedge g \mid \mathrm{ric} \in \mathcal{K}_0(TM)\}$ and that the restriction $\pi^{\mathrm{ric}}: \mathcal{K}_0(TM) \supset \mathcal{W}(TM)^{\perp} \to (T^*M^{\odot 2})_0$ is an isomorphism.

c) Conclude that there are orthogonal decompostions

$$\begin{aligned} \mathcal{K}(TM) &= \mathcal{W}(TM) \oplus ((T^*M^{\odot 2})_0 \owedge g) \oplus (g \owedge g)\mathbb{R} && (1) \\ &= \mathcal{W}(TM) \oplus (T^*M^{\odot 2} \owedge g). && (2) \end{aligned}$$

Remark: Since the Riemann curvature tensor R is a section of $\mathcal{K}(TM) \to M$, it may be decomposed with respect to (1) and (2). In both cases, the $\mathcal{W}(TM)$-part is the *Weyl curvature* $W \in \Gamma(\mathcal{W}(TM))$. The other pieces are the trace-free part of the Ricci curvature and the scalar curvature as well as the Ricci curvature, respectively.