Exercise Sheet no. 7

Exercise 1 (4 points).
Let (M, g) be a Riemannian manifold of dimension n and $L: C^\infty(M) \to C^\infty(M)$ a linear map. Show that the following are equivalent:

(i) L is a partial differential operator of order $\leq k$.

(ii) In any local coordinate system x, defined on $U \subset M$, there exist functions $a_{\alpha_1,\ldots,\alpha_n} \in C^\infty(U)$ for $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, $\alpha_1 + \cdots + \alpha_n \leq k$ such that for all $u \in C^\infty(M)$

$$Lu_{|U} = \sum_{\alpha_1+\cdots+\alpha_n \leq k} a_{\alpha_1,\ldots,\alpha_n} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u.$$

(iii) There is a family $(F_\eta)_{\eta\in I}$ consisting of tuples of at most k vector fields $F_\eta = (X_\eta^1, \ldots, X_\eta^{k_\eta})$ and a smooth function $a \in C^\infty(M)$ such that the collection of supports $(\operatorname{supp}(X_\eta^1) \cap \ldots \cap \operatorname{supp}(X_\eta^{k_\eta}))_{\eta\in I}$ is locally finite and for all $u \in C^\infty(M)$

$$Lu = \sum_{\eta\in I} \partial_{X_\eta^1} \ldots \partial_{X_\eta^{k_\eta}} u + au.$$

Exercise 2 (4 points).
Let (M, g) be a Riemannian manifold and $L: C^\infty(M) \to C^\infty(M)$ a scalar differential operator of order $\leq k$.

a) Show that if L admits a formal adjoint, then it is unique.

b) Consider the scalar differential operator $\partial_X: C^\infty(M) \to C^\infty(M)$ defined by a vector field $X \in \Gamma(TM)$. Show that ∂_X has a formal adjoint and determine it explicitly.
Hint: Calculate $\operatorname{div}(fX)$ for $f \in C^\infty(M)$.

c) Show that any scalar differential operator L has a formal adjoint.
Hint: Characterization (iii) from Exercise 1 might be helpful.

Exercise 3 (4 points).
Let (M, g) be a Riemannian manifold, which is not necessarily complete. Show that for any relatively compact open subset $\Omega \Subset M$, there are sequences $(\tilde{\Omega}_j)_{j\in\mathbb{N}}$ and $(\Omega_j)_{j\in\mathbb{N}}$ of relatively compact open subsets such that

$$\Omega \Subset \tilde{\Omega}_0 \Subset \tilde{\Omega}_1 \Subset \cdots \Subset \bigcup_{j\in\mathbb{N}} \tilde{\Omega}_j \Subset M \qquad \text{and} \qquad \Omega \Subset \bigcap_{j\in\mathbb{N}} \Omega_j \Subset \cdots \Subset \Omega_1 \Subset \Omega_0 \Subset M.$$

Exercise 4 (4 points).
We consider the equation $(\Delta + h)u = f$ on a Riemannian manifold (M, g) for a fixed smooth function $h \in C^\infty(M)$. Assume that $u \in L^q(M)$ for $1 < q < \infty$ and the function f on the right hand side is $C^{k,\alpha}$, for some $k \in \mathbb{N}$ and $0 < \alpha < 1$, with $\|f\|_{C^{k,\alpha}(M)} < \infty$. Show that u has a representative in $C^{k+2,\alpha}$ and for any relatively compact open subset $\Omega \Subset M$ an estimate

$$\|u_{|\Omega}\|_{C^{k+2,\alpha}(\Omega)} \leq C \cdot \left(\|f\|_{C^{k,\alpha}(M)} + \|u\|_{L^q(M)}\right)$$

holds, where the constant $C \in \mathbb{R}$ is independent of u and f.