Geometric PDEs on Manifolds: Exercises

University of Regensburg, Winter term 2022/23
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Please hand in the exercises until **Friday, December 2nd**

Exercise Sheet no. 6

Exercise 1 (4 points).
Consider the manifold with boundary $M = S^1 \times \mathbb{R}_{\geq 0}$ together with its standard volume element dvol. Construct vector fields $X \in \Gamma(T(S^1 \times \mathbb{R}_{\geq 0}))$ such that their divergence $\operatorname{div}(X) = -\delta(X^\flat)$ has compact support and such that:

a) $\displaystyle\int_{S^1\times\mathbb{R}_{\geq 0}} \operatorname{div}(X)\,\mathrm{dvol} = 0$ and $\displaystyle\int_{S^1\times\{0\}} X \lrcorner\, \mathrm{dvol} \neq 0.$

b) $\displaystyle\int_{S^1\times\mathbb{R}_{\geq 0}} \operatorname{div}(X)\,\mathrm{dvol} \neq 0$ and $\displaystyle\int_{S^1\times\{0\}} X \lrcorner\, \mathrm{dvol} = 0.$

Why do these examples not contradict the divergence theorem of Gauß?

Exercise 2 (4 points).
Let (M, g) be a Riemannian manifold, $p \in M$ and $V, W \in T_pM$. Consider the Jacobi field $J_W(t) \coloneqq \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \exp_p((V + sW)t)$ along the geodesic $c(t) = \exp_p(tV)$. Recall that J_W is subject to the Jacobi field equation $\frac{\nabla^2}{\mathrm{d}t^2} J_W(t) + R(J_W(t), \dot c(t))\dot c(t) = 0$ and the pull-back connection satisfies:

- $\frac{\mathrm{d}}{\mathrm{d}t} g(X(t), Y(t)) = g(\frac{\nabla}{\mathrm{d}t}X(t), Y(t)) + g(X(t), \frac{\nabla}{\mathrm{d}t}Y(t))$ for any vector fields X, Y along c.
- $\frac{\nabla}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} c(s,t) = \frac{\nabla}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t} c(s,t)$, where $c\colon U \to M$, $U \subset \mathbb{R}^2$ open, and the pull-back connections differentiate along the curves $t \mapsto c(s,t)$ and $s \mapsto c(s,t)$, respectively.

Show that

$$g(J_W(t), J_W(t)) = t^2 g(W,W) - \frac{1}{3} g(R(W,V)V, W)t^4 + \mathcal{O}(t^5).$$

Exercise 3 (4 points).
Let (M, g) be a compact n-dimensional Riemannian manifold. For a continuous function $w\colon M \to \mathbb{R}$ and some $p \in [2, \frac{2n}{n-2}]$, we define the functional

$$\mathcal{F}(u) = \frac{\int_M (|\nabla u|^2 + wu^2)\,\mathrm{dvol}}{\|u\|_{L^p}^2},$$

where $u \in H^{1,2}(M) \setminus \{0\}$.

a) Show that the functions u under consideration are in L^p, thus verifying well-definedness of $\mathcal{F}$. Also check that $\mathcal{F}$ is invariant under rescaling u by a parameter λ.

b) Prove that $\inf \mathcal{F} > -\infty$ in general and $\inf \mathcal{F} > 0$ if $w > 0$.

Assume in addition that $p < \frac{2n}{n-2}$.

c) Show that $\mathcal{F}$ is weakly lower-semicontinuous, i. e. for every weakly convergent sequence $u_n \rightharpoonup u$ in $H^{1,2}(M) \setminus \{0\}$

$$\liminf_{n\to\infty} \mathcal{F}(u_n) \geq \mathcal{F}(u).$$

Hint: You may use without proof that weakly convergent sequences are bounded (e.g. Werner, Funktionalanalysis, 4. Auflage, Korollar IV.2.3).

d) Prove that there is some $u_0 \in H^{1,2}(M) \setminus \{0\}$ such that $\mathcal{F}(u_0) = \inf \mathcal{F}$.
Hint: You may use without proof that in reflexive Banach spaces (such as the Hilbert space $H^{1,2}(M)$) norm balls are weakly compact, i. e. bounded sequences have weakly convergent subsequences.

Exercise 4 (4 points).
Let (M, g) be a compact n-dimensional Riemannian manifold. For a continuous function $w\colon M \to \mathbb{R}$ and some $p \in [2, \frac{2n}{n-2}]$, we again consider the functional

$$\mathcal{F}(u) = \frac{\int_M (|\nabla u|^2 + wu^2)\,\mathrm{dvol}}{\|u\|_{L^p}^2}$$

defined for all $u \in H^{1,2}(M) \setminus \{0\}$. Show that for $u \in C^\infty(M) \setminus \{0\}$, the following are equivalent:

(i) For all $v \in C^\infty(M)$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{F}(u + tv) = 0.$$

(ii) There is a constant $c \in \mathbb{R}$ with

$$\Delta u + wu = c|u|^{p-2}u.$$

Additionally, if one and hence both of the equivalent conditions are satisfied, express the constant c in terms of $\mathcal{F}(u)$ and $\|u\|_{L^p}$.