Geometric PDEs on Manifolds: Exercises

University of Regensburg, Winter term 2022/23
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Please hand in the exercises until **FRIDAY, November 25th**

Exercise Sheet no. 5

Exercise 1 (4 points).
Let X, Y be metric spaces, $f\colon X \to Y$ a continuous map and Y be complete. Show that the following are equivalent:

(i) $f\colon X \to Y$ is compact.

(ii) $\overline{f(B)}$ is compact for all bounded subsets $B \subset X$.

(iii) For every bounded sequence $(x_k)_{k\in\mathbb{N}}$ in X the sequence $(f(x_k))_{k\in\mathbb{N}}$ has a convergent subsequence.

Exercise 2 (4 points).
Let (M, g) be a compact Riemannian manifold.

a) Show that for any measurable subset $A \subset M$ there exists a measurable subset $B \subset A$ such that $\operatorname{vol}(B) = \operatorname{vol}(A \setminus B)$.

b) Construct a sequence of measurable subsets $(A_k)_{k\in\mathbb{N}}$ of M such that $\operatorname{vol}(A_k \cap A_l) = \operatorname{vol}(A_k \cap A_l^c) = \operatorname{vol}(A_k^c \cap A_l) = \operatorname{vol}(A_k^c \cap A_l^c)$ holds for all $k \neq l \in \mathbb{N}$.

c) Show that the continuous embedding $L^\infty(M) \hookrightarrow L^1(M)$ is not compact.

d) Conclude that for any $p, q \in [1, \infty]$ with $p \geq q$ the embedding $L^p(M) \hookrightarrow L^q(M)$ is not compact.

Exercise 3 (4 points).
Consider a Riemannian manifold (M, g) with its Levi-Civita connection ∇ and its volume form $\mathrm{dvol} \in \Omega^n(M)$. Let $\alpha \in \Omega^1(M)$. Show that there exists a smooth function $\delta(\alpha) \in C^\infty(M)$, called *divergence* of α, that is uniquely determined by either of the following three conditions:

(i) $\mathrm{d}(\alpha^\sharp \lrcorner \,\mathrm{dvol}) = -\delta(\alpha)\,\mathrm{dvol}$, where $\alpha^\sharp \lrcorner\,\mathrm{dvol} \coloneqq \mathrm{dvol}(\alpha^\sharp, -, \dots, -)$ denotes insertion.

(ii) $\delta(\alpha) = -\frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}\left(\sqrt{\det(g)} \cdot g^{ij}\alpha_j\right)$ with $\det(g) = \det((g_{kl})_{k,l=1\dots n})$ in all local coordinates.

(iii) $\delta(\alpha) = -\sum_{i=1}^n (\nabla_{e_i}\alpha)(e_i)$ for all local orthonormal frames $(e_i)_{i=1\dots n}$.

Exercise 4 (4 points).
The theorem of Stone-Weierstraß implies[1] that the set of polynomial functions on an interval $[a, b]$ is dense in $C^0([a, b])$. Here, as it is usual, the space of continuous functions carries the supremum norm. You may use this theorem in the following.

a) Show that polynomial functions on $[-1, 1]^n$ are dense in $C^0([-1, 1]^n)$ for any $n \in \mathbb{N}$.
Hint: You may start to prove the following: Let $p_i\colon [-1, 1]^n \longrightarrow [-1, 1]$ be the i-th canonical projection. Then the image of $C^0([-1, 1])^{\otimes n} \longrightarrow C^0([-1, 1]^n)$, $f_1 \otimes \dots \otimes f_n \mapsto p_1^* f_1 \cdot \ldots \cdot p_n^* f_n$ is dense.

[1] or states; depending on the version

b) Let $n \in \mathbb{N}$. Show that $\tilde{\mathcal{P}}$ from exercise 4 on sheet 4 is dense in $C^0(S^n)$. Conclude that the eigenfunctions of $\Delta^{S^n}: C^\infty(S^n) \longrightarrow C^\infty(S^n)$ span a dense subspace.

Which of these statements remain true, when we equip the spaces of continuous functions with the L^2-norm instead?