Geometric PDEs on Manifolds: Exercises

University of Regensburg, Winter term 2022/23
Prof. Dr. Bernd Ammann, Jonathan Glöckle
Please hand in the exercises until **Tuesday, November 15th**

Exercise Sheet no. 4

Exercise 1 (4 points).
Let (M, d) be a compact metric space and $\mathcal{A} \subset C(M)$ a set of continuous functions. Show that $\mathcal{A}$ is equicontinuous if and only if it is uniformly equicontinuous.

Exercise 2 (4 points).
Let (M, g) be a Riemannian manifold and $p \in M$ a point. We consider Riemannian normal coordinates $\phi: U \to V \subset \mathbb{R}^n$ centered at p (i. e. $\phi(p) = 0$).

a) For any $v_0 \in \mathbb{R}^n$, let v be the constant vector field on V with $v(x) = v_0$ for all $x \in V$. Show that $X := \phi^*(v) \in \Gamma(TU)$ satisfies $\nabla_X X(p) = 0$.

b) Consider the Taylor expansions of the metric coefficients at 0. Deduce that their first order terms vanish, i. e. for all i and j

$$g_{ij}(x) = \delta_{ij} + \mathcal{O}(|x|^2).$$

Hint: Exercise 1 on sheet 3.

Exercise 3 (4 points).
Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. For parameters $k, l \in \mathbb{N}_0$, $p, q \in [1, \infty]$ with $k + l - \frac{n}{q} < k - \frac{n}{p}$ show that the identity on $C^\infty(T^n)$ does not extend to a continuous embedding $H^{k+l,q}(T^n) \to H^{k,p}(T^n)$.
Hint: Look at rescalings of functions as in exercise 2 on sheet 2.

Exercise 4 (4 points).
Let, as in exercise 4 on sheet 2, $\mathcal{P}_k \subset C^\infty(\mathbb{R}^{n+1})$ be the vector space of homogeneous polynomial functions of degree k and $\mathcal{H}_k \subset \mathcal{P}_k$ the subspace of harmonic ones. We write $r^2 = x_1^2 + \cdots x_{n+1}^2$.

a) Argue that $r^2 p \in \mathcal{P}_{k+2}$ for all $p \in \mathcal{P}_k$ and calculate $\Delta(r^2 p)$ in terms of p and Δp.

b) Let $j \in \mathbb{N}_0$ and $p \in \mathcal{H}_{k-2-2j}$. Show that $r^{2j} p$ lies in the image of $\Delta: \mathcal{P}_k \to \mathcal{P}_{k-2}$.

c) Conclude via induction that $\mathcal{P}_k = \mathcal{H}_k \oplus r^2\mathcal{H}_{k-2} \oplus r^4\mathcal{H}_{k-4} \oplus \cdots$ and that

$$0 \longrightarrow \mathcal{H}_k \longrightarrow \mathcal{P}_k \xrightarrow{\Delta} \mathcal{P}_{k-2} \longrightarrow 0$$

is short exact.

d) We denote by $\tilde{\mathcal{P}}_k$ the image of the (injective) restriction map $\mathcal{P}_k \to C^\infty(S^n)$, $p \mapsto p_{|S^n}$. Show that $\tilde{\mathcal{P}}_k \subset \tilde{\mathcal{P}}_{k+2}$ and $\tilde{\mathcal{P}}_k \cap \tilde{\mathcal{P}}_{k+1} = 0$ for all $k \in \mathbb{N}_0$. Deduce that

$$\tilde{\mathcal{P}} := \sum_{k \in \mathbb{N}_0} \tilde{\mathcal{P}}_k = \bigcup_{k' \in \mathbb{N}_0} (\tilde{\mathcal{P}}_{2k'} \oplus \tilde{\mathcal{P}}_{2k'+1}) \subset C^\infty(S^n).$$

Hint: Have a look at the reflection $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $v \mapsto -v$.

e) Conclude that $\tilde{\mathcal{P}} \subset C^\infty(S^n)$ decomposes into a direct sum of eigenspaces of $\Delta^{S^n}: \tilde{\mathcal{P}} \to C^\infty(S^n)$. Determine the multiplicities of the occuring eigenvalues.