Exercise Sheet no. 3

Exercise 1 (4 points).
Let (M,g) be a Riemannian manifold and $x\colon U \to V \subseteq \mathbb{R}^n$ be a chart defined on an open subset $U \subset M$. We denote by g_{ij} the metric coefficients and by ${\Gamma_{ij}}^k$ the Christoffel symbols both with respect to the chart x. For fixed $p \in V$ show that $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$ for all i,j,k if and only if ${\Gamma_{ij}}^k(p) = 0$ for all i,j,k.

Exercise 2 (4 points).
Let $(V, \|\cdot\|_V)$ be a normed vector space and $A \subset V$ a closed subspace. For $x \in V$, set $N(x) \coloneqq \inf_{a\in A} \|x+a\|_V$. Show the following:

a) $N(x) = 0$ if and only if $x \in A$.

b) $[x] \mapsto N(x)$ defines a norm on V/A with the property that the quotient map $V \to V/A$ is continuous.

c) $\operatorname{span}(v) \subset V$ is closed for any $v \in V$.

Assume now that V is complete, $X \subset V$ is a dense subspace and $v \in V \setminus X$. Set $W \coloneqq V/\operatorname{span}(v)$ and denote by $\|\cdot\|_W$ its norm constructed in the first part of the exercise.

d) Show that the canonical map $X \hookrightarrow V \to W$ is injective. Thus X can be identified with a subset of W.

e) Prove that the identity map $(X, \|\cdot\|_V) \to (X, \|\cdot\|_W)$ is continuous and injective, yet its completion $\overline{X}^{\|\cdot\|_V} \to \overline{X}^{\|\cdot\|_W}$ is not injective.

Exercise 3 (4 points).
Prove that the stereographic projection

$$\begin{aligned} S^m \setminus \{(0,\cdots,0,1)^T\} &\longrightarrow \mathbb{R}^m, \\ \begin{pmatrix} \sin(\theta)y \\ \cos\theta \end{pmatrix} &\longmapsto \cot\left(\frac{\theta}{2}\right) y, \end{aligned}$$

where $\theta \in (0,\pi]$ and $y \in S^{m-1}$, is a conformal diffeomorphism.

Exercise 4 (4 points).
Let (M,g) be a complete, connected Riemannian manifold. We want to show that $\mathring{H}^{1,p}(M) = H^{1,p}(M)$ for $p \in [1,\infty)$. To this aim, let d_x be a smooth function on M that is close to the distance function $d(x,\cdot)$ from a point $x \in M$ in the following sense:

$$\|d_x - d(x,\cdot)\|_\infty < 1 \qquad \text{and} \qquad \|\nabla d_x\| < 2.$$

Furthermore, we choose a smooth cut-off function $f\colon \mathbb{R} \to [0,1]$ with $f(t) = 1$ for $t \le 0$, $f(t) = 0$ for $t \ge 1$ and $|f'(t)| < 2$ for all $t \in \mathbb{R}$.

a) Let $u \in H^{1,p}(M)$ be smooth and set $u_j(y) = u(y)f(d_x(y) - j)$ for $j \in \mathbb{N}$. Show that $u_j \in C_c^\infty(M)$ for all j and that $u_j \to u$ for $j \to \infty$ in $H^{1,p}$-norm.

b) Conclude that $\mathring{H}^{1,p}(M) = H^{1,p}(M)$.

c) Give examples showing that in general $\mathring{H}^{1,p}(M) \subsetneqq H^{1,p}(M)$, both in the case $p = \infty$ and in the case where M is not complete.