Geometric PDEs on Manifolds: Exercises

University of Regensburg, Winter term 2022/23
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Please hand in the exercises until
Wednesday, November 2nd, 12:00 noon
Please put them in Box no. 43

Exercise Sheet no. 2

Exercise 1 (4 points).
Let (M, μ) be a measure space. Show the following:

a) Let $p, q, r \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then $\|uv\|_{L^r} \leq \|u\|_{L^p} \|v\|_{L^q}$ for all $u \in L^p(M, \mu)$, $v \in L^q(M, \mu)$.

b) Let $p, q \in [1, \infty]$ with $p \leq q$ and M have finite measure. Then $\|u\|_{L^p} \leq \mu(M)^{\frac{q-p}{pq}} \|u\|_{L^q}$ for all $u \in L^q(M, \mu)$.

c) Let $k \in \mathbb{N}$. Then $\|u_1 \dots u_k\|_{L^1} \leq \|u_1\|_{L^k} \cdots \|u_k\|_{L^k}$ for all $u_1, \dots, u_k \in L^k(M, \mu)$.

Exercise 2 (4 points).
For $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $R \in \mathbb{R}_{>0}$ let u_R be the function defined by $u_R(x) = u(R \cdot x)$ for all $x \in \mathbb{R}^n$. For $k \in \mathbb{N}$ and $p \in [1, \infty]$, show that $\|\nabla^k u_R\|_{L^p} = R^\alpha \|\nabla^k u\|_{L^p}$ for some weight $\alpha \in \mathbb{R}$. Determine α in terms of n, k and p.

Exercise 3 (4 points).
Fill in the details of the proof of the theorem of Gagliardo-Nirenberg (Lemma 1.3) in the case $n = 4$. Alternatively, you may do this in the general case ($n \geq 2$), depending on your preference.

Exercise 4 (4 points).
Let $\mathcal{P}_k \subset C^\infty(\mathbb{R}^{n+1})$ be the space of homogeneous polynomial functions of degree k and $\mathcal{H}_k \subset \mathcal{P}_k$ the subspace of harmonic ones, i. e. the homogeneous polynomial functions p with $\Delta^{g_{eucl}} p = 0$.

a) Determine a basis of $\mathcal{H}_k$ in the cases $k = 0, 1, 2$.

b) Show that the restriction map $\mathcal{H}_k \to C^\infty(S^n), p \mapsto p_{|S^n}$ is injective. Its image will be called $\tilde{\mathcal{H}}_k$.

c) Prove that $\tilde{\mathcal{H}}_k$ consists of eigenfunctions of the Laplace operator Δ on the standard round sphere S^n and determine their eigenvalue.
Hint: Exercise 2 on sheet 1 might help.