Geometric PDEs on Manifolds: Exercises

University of Regensburg, Winter term 2022/23
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Please hand in the exercises until **Tuesday, October 25th**

Exercise Sheet no. 1

Exercise 1 (4 points).
For $\alpha \in \mathbb{Z}^n$ consider the family of complex-valued functions $u_\alpha \coloneqq \exp(2\pi i\langle \alpha, x\rangle)$ on the torus $M = \mathbb{R}^n/\mathbb{Z}^n$. Given a family of complex numbers $(a_\alpha)_{\alpha\in\mathbb{Z}^n}$, show that $\sum_{\alpha\in\mathbb{Z}^n} a_\alpha u_\alpha$ converges in the Sobolev space $H^{k,2}(M)$, $k \in \mathbb{N}$, if and only if $\sum_{\alpha\in\mathbb{Z}^n} |a_\alpha|^2 \|\alpha\|^{2k} < \infty$. Argue furthermore that $H^{k,2}(M) = \{\sum_{\alpha\in\mathbb{Z}^n} a_\alpha u_\alpha \mid \sum_{\alpha\in\mathbb{Z}^n} |a_\alpha|^2 \|\alpha\|^{2k} < \infty\}$.
Hint: You may use Parseval's identity.

Exercise 2 (4 points).
Let (M, g) be a Riemannian manifold of dimension m and ∇ its Levi-Civita connection. For a function $f \in C^\infty(M)$ and two vector fields $X, Y \in \Gamma(TM)$, we define $\nabla^2 f(X, Y) \coloneqq \nabla_X \nabla_Y f - \nabla_{\nabla_X Y} f$

a) Show that $\nabla^2 f$ is a well-defined symmetric $(0, 2)$-tensor on M (the *Hessian* of f).

The *Laplace operator* of f is defined by $\Delta f = -\operatorname{tr}(\nabla^2 f)$. Suppose that $(M, g) \subset (N, h)$ as a hypersurface with unit normal field ν. Here, g is the metric induced by h. The mean curvature of M in N is $H = \frac{1}{m}\operatorname{tr}(-\nabla^h \nu)$.

b) Show $(\Delta^h f)_{|M} = -(\nabla^{h,2}_{\nu,\nu} f)_{|M} + mH\nabla_\nu f + \Delta^g(f_{|M})$ for all $f \in C^\infty(N)$.