Seminar on semiclassical analysis

Winter term 2022/23

Prof. Bernd Ammann

Time and Place of the seminar: Tuesday 14.15 to 16.00, in Ph 7.1.21

Number of sessions: **10**

Available Dates 15.11., 29.11., 6.12., 13.12., 20.12., 10.1., 17.1., 24.1., 31.1., 7.2.

Distribution of talks: Oct, 31 st.
Special obstructions:

- Nov 22: Bernd and Jonathan not in Regensburg
- Dec 20: Guadalupe not in Regensburg

Content

One goal of the seminar is to prepare a seminar on semiclassical analysis, planned for the summer term. This includes two topics:

- symplectic geometry and classical mechanics,
- pseudodifferential operators.

Another goal is to broaden our knowledge about important techniques.

Preliminaries

We assume that every particpant is familiar with symplectic vector spaces, differential geometry, has some familiarity with Fourier transformation and th theory of distributions.

1 Talks about symplectic geometry

References: Further literature on this subject [7], [12], [3], [2], [11], also [4] and its translation [5].

Talk no. 1: Symplectic manifolds and Lagrangian Submanifolds. *15.11.* Symplectic manifolds [6, Sec. 1.3 and 1.4]. The cotangent space T^*Q as an exact symplectic manifold [6, Chap. 2]. Mention Darboux's theorem (e.g. [6, Theorem 8.1]) without proof, this will be proved later in Talk no. 4. Lagrangian submanifolds in general and in T^*Q, generating functions, conormal bundles and applications to symplectomorphisms [6, Chap. 3].

Talk no. 2: Hamiltonian systems. *29.11.*
Hamiltonian systems [11, Sec. 5.4] or [7, Part A, Sec. 8.1]; please also include some simple examples, such as geodesics on a Riemannian manifold, or more generally systems on T^*Q with Hamilton function $H(x,\xi) = \frac{1}{2}g(\xi,\xi) + V(x)$, where $x \in M$ and $\xi \in T^*_x M$. Poisson brackets and more generally Poisson manifolds [11, Sec. 5.5] or [7, Part A, Chap. 9], including associated Hamiltonian flows, symplectc leaves, Darboux's Theorem for Poisson structures [7, Part A, Chap. 9].

Talk no. 3: Moser Trick. *6.12.*
The Moser trick and its variants [6, Chap. 7]. Recall concepts from Chap 4-6 from this book if needed.

Talk no. 4: Theorems by Darboux and Weinstein. *13.12.*
The main result of this talk is Weinstein's tubular neighborhood theorem [6, Theorem 9.3], which states that a tubular neighborhood of a (compact?) Lagrangian submanifold L is symplectomorphic to a neighborhood of the zero section in T^*L. This is proved using the Moser trick. For pedagogical reasons it might be good to start with a proof of Darboux's theorem in order to illustrate the method of proof. [6, Chap. 8] is then devoted to preparatory steps, the main proof is in [6, Sec. 9.2].

Supplementary Talk no. 1: Moment maps and Symplectic reduction
Moment maps and symplectic reduction [7, Part B, Chap. 11 and 12]

Supplementary Talk no. 2: Convexity and Delzant's Theorem [7, Part B, Chap. 13 and 14]

2 Talks about Pseudodifferential operators

References: Classical books on this subject are [13] and [9, Chap. XVIII]. They are broad and contain a lot, but potentially a bit tedious to read as a start. For the "local" theory, i.e. on $\mathbb{R}^n$, I like e.g. Abel's book [1]. For global aspects on manifolds, the presentation in [10] is very efficient, but contains some small inaccuracies with smoothing operators, which are comparably easy to fix.

Talk no. 5: Fourier transformation. *20.12.*
Fourier transform on the space of Schwartz functions and its dual space. The minimal program for this talk is [14, Sections 3.1 and 3.2]. Obviously, there are many other interesting aspects one may include, see e.g. [1, Chap. 2]. Also include the semiclassical version [14, Sections 3.3].

Talk no. 6: The stationary phase method and oscillatory integrals. *10.1.*
Discuss the method of stationary phase in one and several dimensions, following [14, Sections 3.4 and 3.5]. Additional reference [8, Sec. 7.7]. Also explain oscillatory integrals [14, Sections 3.6] (see [1, Sec. 3.3] for an alternative reference). Additional reference [8, Sec. 7.8].

Talk no. 7: Kohn-Nirenberg symbols, Weyl quantization, and pseudo-differential operators. *17.1. +24.1.*
Introduce the symbol classes $S^m_{\rho,\delta}$, the Kohn–Nirenberg symbols [13, Chap II, §1, Def. 1.1]. Also introduce classical symbols of order m, which is the class S^m in the language of [13, Chap II, §1]: these are those symbols p, that admit a formal development $p \sim \sum_{j\in\mathbb{N}_0} p_{m-j}$ where p_{m-j} is a homogeneous symbol of homogeneity $m-j$ (away from neighborhood of 0).[1] Then introduce pseudo-differential operators on compact manifolds. The presentation [10, III §3] is a very efficient way to do this, but argue carefully, why the various characterizations of smoothing operators are the same (discuss this with Bernd). The map from symbols to pseudodifferential operators is called quantization, explain why? Also clearly emphasize that the quantization map in [10, III §3] is a bit different from Weyl quantization ($a(x,\xi)$ versus $a((x+y)/2,\xi)$) which is used in [14, Chap. 4, e.g. Subs. 4.1.1].[2] Discuss why Weyl quantization behaves better concerning adjoints, see [14, Subs. 4.1.2 Theorem 4.1]. Introduce and discuss the principal symbol for the cases $S^m_{1,0}$ and S^m_{cl}, and discuss that in the classical case the principal symbol on a Riemannian manifold (M,g) can be seen as an function on the sphere bundle of T^*M.

Talk no. 8: More on symbols, ellipticity and parametrices. *31.1. +7.2.*
Continue with the introduction to pseudodifferential operators and their applications [10, III §3 – §5]. In particular, define the multiplication of symbols for standard quantization [10, Theorem 3.10] and [14, Subs. 4.3.4]. Show or discuss that the commutator of symbols is its Poisson bracket (Reference? Maybe just calculate!). Define ellipticity and construct parametrices and discuss their main properties. Applications to elliptic regularity etc.

Talk no. 9: Quantization from the point of view of semiclassics. *next term* **(or will be skipped)**
Quantization of Schwartz class symbols [14, Sec. 4.1]. Various quantization formulas [14, Sec. 4.2], in particular for exponentials of linear and quadratic symbols. Multiplication of symbols for Weyl quantization, see [14, Sec. 4.3]. Introduce the symbol classes S_δ [14, Sec. 4.4].

Supplementary Talk no. 3: Wave front sets and their distribution
Definition of the wave front set of distributions. Propagation of the wave front set with respect to normally hyperbolic operators

Seminar-Homepage

`https://ammann.app.uni-regensburg.de/lehre/2022w_semiclassic`

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[1] Attention: This defintion of "classical" is slightly stronger than the one in [10, III §13], and the meaning of S^m changes from source to source. Thus I would prefer to write S^m_{cl}.

[2] One might introduce Op_t as in [14] and then $t=1$ leads to the quantization in [10], while $t=1/2$ leads to Weyl quantization.

Literatur

[1] Abels, H. Pseudodifferential and singular integral operators. De Gruyter Graduate Lectures. De Gruyter, Berlin, 2012. An introduction with applications.

[2] Abraham, R., and Marsden, J. Foundations of mechanics. Benjamin/Cummings Publishing Co., Inc., Advanced Book Program, Reading, Mass., 1978. Second edition, revised and enlarged, With the assistance of Tudor Raţiu and Richard Cushman.

[3] Arnol'd, V. Mathematical methods of classical mechanics, second ed., vol. 60 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1989. Translated from the Russian by K. Vogtmann and A. Weinstein.

[4] Berndt, R. Einführung in die symplektische Geometrie. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Braunschweig, 1998.

[5] Berndt, R. An introduction to symplectic geometry, vol. 26 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001. Translated from the 1998 German original by Michael Klucznik.

[6] Cannas da Silva, A. Lectures on symplectic geometry, vol. 1764 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001.

[7] Cieliebak, K. Symplectic geometry. Lecture notesscript (unpublished), 2010. Available at this links: Part A and Part B.

[8] Hörmander, L. The analysis of linear partial differential operators. I, second ed. Springer-Verlag, Berlin, 1990. Distribution theory and Fourier analysis.

[9] Hörmander, L. The analysis of linear partial differential operators. III, Pseudo-Differential Operators, corrected second printing ed. Springer-Verlag, Berlin, 1994.

[10] Lawson, H. B., and Michelsohn, M.-L. Spin Geometry. Princeton University Press, Princeton, 1989.

[11] Marsden, J., and Ratiu, T. Introduction to mechanics and symmetry, second ed., vol. 17 of Texts in Applied Mathematics. Springer-Verlag, New York, 1999. A basic exposition of classical mechanical systems.

[12] McDuff, D., and Salamon, D. Introduction to symplectic topology, third ed. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2017.

[13] Taylor, M. E. Pseudodifferential operators. Princeton University Press, Princeton, N.J., 1981.

[14] Zworski, M. Semiclassical analysis, vol. 138 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2012. DOI: 10.1090/gsm/138.