

Seminar on semiclassical analysis

Winter term 2022/23

Prof. Bernd Ammann

Time and Place of the seminar: Tuesday 14.15 to 16.00, in Ph 7.1.21

Number of sessions: **10**

Available Dates 15.11., 29.11., 6.12., 13.12., 20.12., 10.1., 17.1., 24.1., 31.1., 7.2.

Distribution of talks: Oct, 31 st.

Special obstructions:

- Nov 22: Bernd and Jonathan not in Regensburg
- Dec 20: Guadalupe not in Regensburg

Content

One goal of the seminar is to prepare a seminar on semiclassical analysis, planned for the summer term. This includes two topics:

- symplectic geometry and classical mechanics,
- pseudodifferential operators.

Another goal is to broaden our knowledge about important techniques.

Preliminaries

We assume that every participant is familiar with symplectic vector spaces, differential geometry, has some familiarity with Fourier transformation and the theory of distributions.

1 Talks about symplectic geometry

References: Further literature on this subject [7], [12], [3], [2], [11], also [4] and its translation [5].

Talk no. 1: Symplectic manifolds and Lagrangian Submanifolds. 15.11.
RAPHAEL SCHMIDPETER.

Symplectic manifolds [6, Sec. 1.3 and 1.4]. The cotangent space T^*Q as an exact symplectic manifold [6, Chap. 2]. Mention Darboux's theorem (e.g. [6, Theorem 8.1]) without proof, this will be proved later in Talk no. 4. Lagrangian submanifolds in general and in T^*Q , generating functions, conormal bundles and applications to symplectomorphisms [6, Chap. 3].

Talk no. 2: Hamiltonian systems. *29.11.* BERND AMMANN.

Hamiltonian systems [11, Sec. 5.4] or [7, Part A, Sec. 8.1]; please also include some simple examples, such as geodesics on a Riemannian manifold, or more generally systems on T^*Q with Hamilton function $H(x, \xi) = \frac{1}{2}g(\xi, \xi) + V(x)$, where $x \in M$ and $\xi \in T_x^*M$. Poisson brackets and more generally Poisson manifolds [11, Sec. 5.5] or [7, Part A, Chap. 9], including associated Hamiltonian flows, symplectic leaves, Darboux's Theorem for Poisson structures [7, Part A, Chap. 9].

Talk no. 3: Moser Trick. *6.12.* GUADALUPE CASTILLO SOLANO.

The Moser trick and its variants [6, Chap. 7]. Recall concepts from Chap 4-6 from this book if needed.

Talk no. 4: Theorems by Darboux and Weinstein. *13.12.* JONATHAN GLÖCKLE.

The main result of this talk is Weinstein's tubular neighborhood theorem [6, Theorem 9.3], which states that a tubular neighborhood of a (compact?) Lagrangian submanifold L is symplectomorphic to a neighborhood of the zero section in T^*L . This is proved using the Moser trick. For pedagogical reasons it might be good to start with a proof of Darboux's theorem in order to illustrate the method of proof. [6, Chap. 8] is then devoted to preparatory steps, the main proof is in [6, Sec. 9.2].

Supplementary Talk no. 1: Moment maps and Symplectic reduction
Moment maps and symplectic reduction [7, Part B, Chap. 11 and 12]

Supplementary Talk no. 2: Convexity and Delzant's Theorem [7, Part B, Chap. 13 and 14]

2 Talks about Pseudodifferential operators

References: Classical books on this subject are [13] and [9, Chap. XVIII]. They are broad and contain a lot, but potentially a bit tedious to read as a start. For the "local" theory, i.e. on \mathbb{R}^n , I like e.g. Abel's book [1]. For global aspects on manifolds, the presentation in [10] is very efficient, but contains some small inaccuracies with smoothing operators, which are comparably easy to fix.

Talk no. 5: Fourier transformation. *20.12.* CHRISTOPH KRPOUN.

Fourier transform on the space of Schwartz functions and its dual space. The minimal program for this talk is [14, Sections 3.1 and 3.2]. Obviously, there are many other interesting aspects one may include, see e.g. [1, Chap. 2]. Also include the semiclassical version [14, Sections 3.3].

Talk no. 6: The stationary phase method and oscillatory integrals.
10.1. JULIAN SEIPEL.

Discuss the method of stationary phase in one and several dimensions, following [14, Sections 3.4 and 3.5]. Additional reference [8, Sec. 7.7]. Also explain oscillatory integrals [14, Sections 3.6] (see [1, Sec. 3.3] for an alternative reference). Additional reference [8, Sec. 7.8].

Talk no. 7: Kohn-Nirenberg symbols, Weyl quantization, and pseudo-differential operators. 17.1. ROMAN SCHIESSL.

Introduce the symbol classes $S_{\rho,\delta}^m$, the Kohn–Nirenberg symbols [13, Chap II, §1, Def. 1.1]. Also introduce classical symbols of order m , which is the class S^m in the language of [13, Chap II, §1]: these are those symbols p , that admit a formal development $p \sim \sum_{j \in \mathbb{N}_0} p_{m-j}$ where p_{m-j} is a homogeneous symbol of homogeneity $m-j$ (away from neighborhood of 0).¹ Then introduce pseudo-differential operators on compact manifolds. The presentation [10, III §3] is a very efficient way to do this, but argue carefully, why the various characterizations of smoothing operators are the same (discuss this with Bernd). The map from symbols to pseudodifferential operators is called quantization, explain why? Also clearly emphasize that the quantization map in [10, III §3] is a bit different from Weyl quantization ($a(x, \xi)$ versus $a((x+y)/2, \xi)$) which is used in [14, Chap. 4, e.g. Subs. 4.1.1].² Discuss why Weyl quantization behaves better concerning adjoints, see [14, Subs. 4.1.2 Theorem 4.1]. Introduce and discuss the principal symbol for the cases $S_{1,0}^m$ and S_{cl}^m , and discuss that in the classical case the principal symbol on a Riemannian manifold (M, g) can be seen as a function on the sphere bundle of T^*M .

Talk no. 8: More on symbols, ellipticity and parametrices. 24.1. MATTIAS LUDEWIG.

Continue with the introduction to pseudodifferential operators and their applications [10, III §3 – §5]. In particular, define the multiplication of symbols for standard quantization [10, Theorem 3.10] and [14, Subs. 4.3.4]. Show or discuss that the commutator of symbols is its Poisson bracket (Reference? Maybe just calculate!). Define ellipticity and construct parametrices and discuss their main properties. Applications to elliptic regularity etc.

Talk no. 9: Quantization from the point of view of semiclassics. 31.1. N.N..

Quantization of Schwartz class symbols [14, Sec. 4.1]. Various quantization formulas [14, Sec. 4.2], in particular for exponentials of linear and quadratic symbols. Multiplication of symbols for Weyl quantization, see [14, Sec. 4.3]. Introduce the symbol classes S_δ [14, Sec. 4.4].

Supplementary Talk no. 3: Wave front sets and their distribution

Definition of the wave front set of distributions. Propagation of the wave front set with respect to normally hyperbolic operators

Seminar-Homepage

https://ammann.app.uni-regensburg.de/lehre/2022w_semiclassic

¹Attention: This definition of “classical” is slightly stronger than the one in [10, III §13], and the meaning of S^m changes from source to source. Thus I would prefer to write S_{cl}^m .

²One might introduce Op_t as in [14] and then $t = 1$ leads to the quantization in [10], while $t = 1/2$ leads to Weyl quantization.

Literatur

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