Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
Prof. Dr. Bernd Ammann, Jonathan Glöckle
Please hand in the exercises until **Friday, January 28**

Exercises Sheet no. 12

1. Exercise (4 points).

a) Show that there is a diffeomorphism $\mathrm{SO}(3) \cong \mathbb{R}P^3$.
Hint: It is convenient to view $\mathbb{R}P^3$ as quotient of S^3 with respect to the antipodal action.

b) Prove that

$$\left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \middle| \, z, w \in \mathbb{C} \right\} \longrightarrow \mathbb{H}$$
$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \longmapsto z + wj$$

defines a ring isomorphism.

c) Show that the ring isomorphism from b) restricts to an isomorphism $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$ of Lie groups.

2. Exercise (4 points).
For $n \in \mathbb{N}$, we denote by $\chi\colon \mathrm{Spin}(n) \to \mathrm{SO}(n)$ the canonical homomorphism from the spin group $\mathrm{Spin}(n) \subset \mathrm{Cl}_n$ to the special orthogonal group in dimension n.

a) Determine the Lie algebra $\mathfrak{spin}(n) = T_{\mathbb{1}}\mathrm{Spin}(n)$ as subset of Cl_n. (Use the definition of $\mathrm{Spin}(n)$ given in Def. 4.4.14 of the script. You may use the fact that $\mathrm{Spin}(n)$ is a Lie group, and that χ is a local diffeomorphism and a Lie group epimorphism whose kernel has two elements.)

b) Calculate the induced Lie algebra homomorphism $\mathrm{d}_{\mathbb{1}}\chi\colon \mathfrak{spin}(n) \to \mathfrak{so}(n)$.
Hint: It might be useful to introduce the notation $e_i \wedge e_j$ for the matrix in $\mathfrak{so}(n) \subset \mathbb{R}^{n\times n}$ with 1 in the j-th row of the i-th column, -1 in the i-th row of the j-th column and 0 elsewhere.

3. Exercise (2 points).
Construct a spin structure on the sphere $\mathbb{S}^n$.
Hint: Identify $P_{\mathrm{SO}}(\mathbb{S}^n)$ *with* $\mathrm{SO}(n+1)$ *and use* $\mathrm{Spin}(n+1)$.

4. Exercise (4 points).

a) Argue why S^n is simply connected for $n \geq 2$.

b) Let $p : X \to S^n$ be a submersion, with X compact, $n \geq 2$. Assume that every preimage $p^{-1}(x)$, $x \in S^n$ is diffeomorphic to Y. Show that if Y is simply-connected, then X is simply-connected as well. You may use the fact that for any commuting square of smooth maps

$$\begin{array}{ccc} S^1 \times \{0\} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow p \\ S^1 \times [0,1] & \longrightarrow & S^n \end{array}$$

there is a lift $S^1 \times [0,1] \longrightarrow X$.

c) Show by induction over n that $\mathrm{Spin}(n)$ is simply-connected for $n \geq 3$.
Hint: use the previous exercise.