Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
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Please hand in the exercises until **Friday, January 21**

Exercises Sheet no. 11

1. Exercise (4 points).

a) Let $(V, \langle \bullet, \bullet \rangle) = \mathbb{R}^{n,1}$ be the $(n+1)$-dimensional Minkowski space. Show that for all $\alpha \in V^*$ and for $t = -\frac{\langle \alpha, \alpha \rangle}{2}$

$$T = \alpha \otimes \alpha + t \langle \bullet, \bullet \rangle$$

satisfies the dominant energy condition.
Hint: show it in the special cases that α is timelike resp. spacelike, and conclude the lightlike case by a density argument.

b) Show that T defined as above satisfies the dominant energy condition if $t \le -\frac{\langle \alpha, \alpha \rangle}{2}$

c) Let (M, g) be a Lorentzian manifold and $\phi \in C^\infty(M, \mathbb{R})$. Show that that the energy-momentum T_ϕ of ϕ – interpreted as a scalar field and as given in Conclusion 1.3.1 of this this lecture's script –

$$T_\phi := \frac{1}{2}(\mathrm{d}\phi \otimes \mathrm{d}\phi) - \frac{1}{4}\langle \mathrm{d}\phi, \mathrm{d}\phi \rangle_g g - \frac{m^2}{4} |\phi|^2 g$$

satisfies the dominant energy condition.

d) (Bonus exercise) Show that the tensor T defined in a) never satisfied the dominant energy condition for $t > -\frac{\langle \alpha, \alpha \rangle}{2}$.
Hint: Give counterexamples for α timelike, spacelike and lightlike.

2. Exercise (4 points).
The goal of this exercise is to calculate the ADM mass of the standard spacelike hypersurface in the Schwarzschild metric, which is given by the Riemannian metric

$$g(r, x) = \frac{1}{1 - \frac{2m}{r^{n-2}}} \mathrm{d}r^2 + r^2 g_{S^{n-1}}(x)$$

on $((2m)^{1/(n-2)}, \infty) \times S^{n-1}$.

a) Consider the function $r(x^1, \dots, x^n) = \sqrt{(x^1)^2 + \dots (x^n)^2}$ on $(\mathbb{R}^n \setminus \{0\}, g_{eucl})$. Establish that $\nabla_{\mathrm{grad}\, r} \mathrm{d}r = 0$ and $\Delta r = \frac{1-n}{r}$.

b) With respect to the chart $\phi \colon ((2m)^{1/(n-2)}, \infty) \times S^{n-1} \to \mathbb{R}^n \setminus \overline{B_{(2m)^{1/(n-2)}}(0)}$, $(r, x) \mapsto rx$, determine $\eta = (\phi^{-1})^* g - g_{eucl}$ and $\mathrm{d}\,\mathrm{tr}(\eta) - \mathrm{div}\, \eta$.

c) Verify that the ADM-mass is m.

3. Exercise (4 points).
We continue with Exercise 3 from Exercise sheet no. 10. Let (M, g) be an n-dimensional Riemannian manifold, $n \geq 3$. We assume that X is a conformal vector field, and we write $(\phi_t^X)^* g = u_t^{4/(n-2)} g$.

a) Show
$$\Delta^g(\delta X^\flat) = \frac{1}{n-1} \operatorname{scal}^g \delta X^\flat - \frac{n}{2(n-1)} \partial_X \operatorname{scal}^g .$$
Hint (in the notation of the previous exercise): Use that for $\tilde{g} = u^{4/(n-2)} g$ *we have* $\operatorname{scal}^{\tilde{g}} = F(u)$ *and then calculate* $(d/dt)|_{t=0} \operatorname{scal}^{(\phi_t^X)^* g}$ *in two ways.*

b) Show, that if M is compact, then
$$\int_M \partial_X \operatorname{scal}^g \operatorname{dvol}^g = 0.$$

Now, let g_{sph} be the standard metric on S^n, and $x^0, x^1, \ldots, x^n$ the restriction of the coordinate functions of $\mathbb{R}^{n+1}$ to S^n. You may recall (without proof) from Exercise 3 of Sheet no. 7, that $X := \operatorname{grad} x^0 \in \Gamma(S^n)$, where the gradient is taken on (S^n, g_{sph}), is a conformal vector field of (S^n, g_{sph}).

c) Show that a function of the form $a + bx^0$, $a, b \in \mathbb{R}$, $b \neq 0$, cannot be the scalar curvature of a metric conformal to g_{sph}.