# Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
Prof. Dr. Bernd Ammann, Jonathan Glöckle
Please hand in the exercises until **Friday, January 14**

**Exercises Sheet no. 10**

**1. Exercise** (4 points).
Consider the Yamabe functional

$$\mathcal{E}_q(u) = \tfrac{4(n-1)}{n-2} \frac{\int_{\mathbb{R}^n} u\Delta u \,\mathrm{dvol}}{\|u\|_{L^q}^2}$$

of flat Euclidean space. Let $u \in C_c^\infty(\mathbb{R}^n)$ be a compactly supported smooth function and set $u_\epsilon(x) = u(\frac{x}{\epsilon})$ for $\epsilon > 0$.

a) Depending on the value of $q \in [1, \infty]$, determine the limiting behavior of $\mathcal{E}_q(u_\epsilon)$ for $\epsilon \to 0$ and $\epsilon \to \infty$.

b) In the case $q = \frac{2n}{n-2}$, show that $\mathcal{E}_q(u) \geq n(n-1)\omega_n^{n/2}$.
*Hint: Use the stereographic projection to construct a related test function* $\hat{u} \in C_c^\infty(S^n \smallsetminus \{\infty\})$.

**2. Exercise** (4 points).
Let $M$ be a compact $n-1$-dimensional manifold. We consider $M \times S^1$ with the $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$-action given by multiplication in the second component. As in Exercise 1 on sheet 6 we restrict our attention to $S^1$-invariant objects. For example, the Yamabe functional on $S^1$-invariant functions $u \in C^\infty(M \times S^1)^{S^1}$ (which we identify with functions in $C^\infty(M)$) is given by

$$\mathcal{E}_{p_n}(u) = \frac{\int_{M\times S^1} \left(a_n |\mathrm{d}u|_h^2 + \mathrm{scal}^h u^2\right) \mathrm{dvol}^h}{\|u\|_{L^{p_n}(M\times S^1)}^2}$$

for fixed $h \in \mathcal{R}(M \times S^1)^{S^1}$. Here, $p_n = \frac{2n}{n-2}$ and $a_n = 4\frac{n-1}{n-2}$ as usual.

a) Existence of minimizers of this functional is easier to establish than in the usual Yamabe problem. The reason is that the $S^1$-invariant problem is subcritical. Make this plausible in the case $h = g + L\mathrm{d}t^2$, $g \in \mathcal{R}(M)$ and $L \in \mathbb{R}_{>0}$, by checking that $p_n < p_{n-1}$ and rewriting $\mathcal{E}_{p_n}(u)$ so that integration is performed over $(M, g)$ only.

b) Let $h_0$ be a minimizer of the volume-normalized Einstein-Hilbert functional $\mathcal{E}$ in the $S^1$-invariant conformal class $[h]^{S^1}$. Show that $h_0$ has constant scalar curvature.

c) Suppose that $g \in \mathcal{R}(M)$ with $\mathrm{scal}^g > 0$ and let $h = g + L\mathrm{d}t^2$. Prove that there is a constant $C > 0$ such that for any $u \in C^\infty(M \times S^1)^{S^1}$

$$\mathcal{E}_{p_n}(u) \geq \frac{\min(a_n, \mathrm{scal}^g)}{C^2} L^{2/n}.$$

d) Conclude that in the situation of part c) and for large $L$, the minimizer from part b) is not a minimizer of $\mathcal{E}$ within the whole conformal class $[h]$.

**3. Exercise** (6 points).
Let $(M,g)$ be an $n$-dimensional Riemannian manifold, $n \geq 3$. Assume that the flow $\phi_t^X : M \to M$ of a vector field $X \in \Gamma(TM)$ exists for all $t \in \mathbb{R}$.

a) Show that $X$ is conformal (i.e. $\phi_t^X$ is a conformal diffeomorphism for all $t$) if and only if

$$\mathcal{L}_X g = -\frac{2}{n}(\delta X^\flat)\, g\,.$$

From now on we assume that $X$ is a conformal vector field, and we write $(\phi_t^X)^* g = u_t^{4/(n-2)} g$.

b) Show that

$$\frac{d}{dt}\bigg|_{t=0} u_t = -\frac{n-2}{2n}\delta X^\flat.$$

c) We define $S : C^\infty(M, \mathbb{R}_+) \to C^\infty(M)$ as $S(u) := u^{-(n+2)/(n-2)} L^g u$. Calculate the derivative $S_1' : C^\infty(M) \to C^\infty(M)$ of $S$ at $u \equiv 1$.

*The remaining parts of this exercise will be shifted to Exercise sheet no. 11.*

d) Show

$$\Delta^g(\delta X^\flat) = \frac{1}{n-1}\,\mathrm{scal}^g\,\delta X^\flat - \frac{n}{2(n-1)}\partial_X\,\mathrm{scal}^g\,.$$

*Hint: Use that for* $\tilde{g} = u^{4/(n-2)} g$ *we have* $\mathrm{scal}^{\tilde{g}} = F(u)$ *and then calculate* $(d/dt)|_{t=0}\,\mathrm{scal}^{(\phi_t^X)^* g}$ *in two ways.*

e) Show, that if $M$ is compact, then

$$\int_M \partial_X\,\mathrm{scal}^g\,\mathrm{dvol}^g = 0.$$

*Note: This equation will be used later on to prove that there are "many" functions* $f : S^n \to \mathbb{R}$ *that cannot be achieved as scalar curvature by a conformal change of a metric.*