Differential Geometry III: Exercises

University of Regensburg, Winter term 2021/22
Prof. Dr. Bernd Ammann, Jonathan Glöckle
Please hand in the exercises until **Friday, January 7**
More precisely: either send an electronic version to Jonathan until Jan 7, or bring a paper version to the lecture on Jan 10.

Exercises Sheet no. 9

1. Exercise (4 points).
Let (M, d_m) and (N, d_N) be metric spaces and $\overline{M}$ and $\overline{N}$ their respective completions. Given a uniformly continuous map $f\colon M \to N$ show that there is a unique uniformly continuous map $\overline{f}\colon \overline{M} \to \overline{N}$ such that

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \big\downarrow & & \big\downarrow \\ \overline{M} & \xrightarrow{\overline{f}} & \overline{N} \end{array}$$

commutes.

2. Exercise (4 points).
Let $(V, \|\cdot\|_V)$ be a normed vector space and $A \subset V$ a closed subspace. For $x \in V$, set $N(x) \coloneqq \inf_{a\in A} \|x + a\|_V$. Show the following:

a) $N(x) = 0$ if and only if $x \in A$.

b) $[x] \mapsto N(x)$ defines a norm on V/A with the property that the quotient map $V \to V/A$ is continuous.

c) $\operatorname{span}(v) \subset V$ is closed for any $v \in V$.

Assume now that V is complete, $X \subset V$ is a dense subspace and $v \in V \setminus X$. Set $W \coloneqq V/\operatorname{span}(v)$ and denote by $\|\cdot\|_W$ its norm constructed in the first part of the exercise.

d) Show that the canonical map $X \hookrightarrow V \to W$ is injective. Thus X can be identified with a subset of W.

e) Prove that the identity map $(X, \|\cdot\|_V) \to (X, \|\cdot\|_W)$ is continuous and injective, yet its completion (in the sense of Exercise 1) is not injective.

3. Exercise (4 points).
For a compact manifold Q of dimension $q \geq 3$ and a Riemannian metric g on Q we define

$$\mathcal{E}^Q(g) \coloneqq \frac{\int_Q \operatorname{scal}^g \operatorname{dvol}^g}{\operatorname{vol}(Q,g)^{(q-2)/q}}.$$

Again, the *conformal Yamabe constant* of a conformal class $[g^Q]$ on Q is defined as

$$Y(Q,[g^Q]) \coloneqq \inf \left\{ \mathcal{E}^Q(g) \mid g \in [g^Q] \right\}.$$

Now, assume that we have two compact Riemannian manifolds (M^m, g^M) and (N^n, g^N) of dimensions $m, n \geq 3$, with $\mathcal{E}^M(g^M) = Y(M,[g^M])$ and $\mathcal{E}^N((g^N) = Y(N,[g^N])$. We write the product metric on $M \times N$ as $g^{M\times N} = (\pi^M)^* g^M + (\pi^N)^* g^N$.

a) Show that $g^{M\times N}$ has constant scalar curvature $s_0 \in \mathbb{R}$. Conclude that it is a stationary point of the Einstein-Hilbert functional on $[g^{M\times N}]$.

b) Show that in the case $s_0 \le 0$, we have

$$\mathcal{E}^{M\times N}(g^{M\times N}) = Y(M\times N, [g^{M\times N}]). \qquad (*)$$

Hint: You may use that a Riemannian metric $\widetilde{g} \in [g^{M\times N}]$ exists with $\mathcal{E}^{M\times N}(\widetilde{g}) = Y(M\times N, [g^{M\times N}])$.

c) Construct and example for M, N, g^M, g^N on which condition $(*)$ does not hold.

Hint: $M = N = S^3$, $g^M = g_{\mathrm{sph}}$, $g^N = L^2 g_{\mathrm{sph}}$, $L \gg 1$.

d) Construct a compact connected Riemannian manifold (Q, g) such that there are at least two metrics g_1 and g_2 in $[g]$ with $\mathrm{vol}(Q, g_1) = \mathrm{vol}(Q, g_2) = 1$ and with both $\mathrm{scal}^{g_1} = s_1$ and $\mathrm{scal}^{g_2} = s_2$ constant, with $s_1 > n(n-1)\omega_n^{2/n} \ge s_2$.

Hint: You may use that a Riemannian metric $\widetilde{g} \in [g]$ exists with $\mathcal{E}^Q(\widetilde{g}) = Y(Q, [g])$.

e) *(Open Problem – 10000 bonus points!)* Let $Y(M, [g^M]) > 0$ and $Y(N, [g^N]) > 0$. Prove or disprove (by counterexample): There is a $\mu \in \mathbb{R}_{>0}$, such that $g_\mu := (\pi^M)^* g^M + \mu \cdot (\pi^N)^* g^N$ satisfies

$$\mathcal{E}^{M\times N}(g_\mu) = Y(M\times N, [g_\mu]).$$

Sorry for the extremely hard bonus exercise; in case you solve the bonus exercise, we are deeply impressed: this will lead to your first publication. Bernd Ammann was only able to prove it up to a constant, i.e. there is a $\mu > 0$ with

$$\frac{a_{n+m}}{{a_n}^{n/(n+m)}{a_m}^{m/(n+m)}}\mathcal{E}^{M\times N}(g_\mu) \le Y(M\times N, [g_\mu]]) \le \mathcal{E}^{M\times N}(g_\mu),$$

see here.

Merry Christmas – and as soon you get back to work – joy with the exercises.